TOPICS IN GEOMETRY

Eric Lehman and Martti Pesonen

revised February 24, 2007

Contents

1 Parallel projections and central projections in plane geometry

A graduation *of a line is a map that associates a real number to each point of the line. A non regular graduation may look like in Figure 1.*
A regular graduation *is the most commonly used on rulers, see Figure 2.*

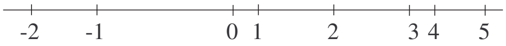

Figure 1: Line with a graduation

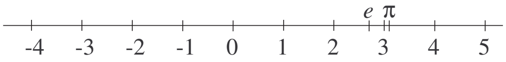

Figure 2: Line with regular graduation

Imagine two lines d and d' and a point S in a plane. Let us supply the line d with a regular graduation, like in Figure 3. From S we draw lines through the marks of the graduation of d and take the intersections with d' (see Figure 4). Do you get a regular graduation on this second line? No! But is there anyway some kind of regularity left? We shall answer this question.

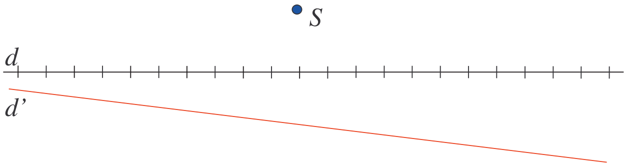

Figure 3: Line with regular graduation and a point outside

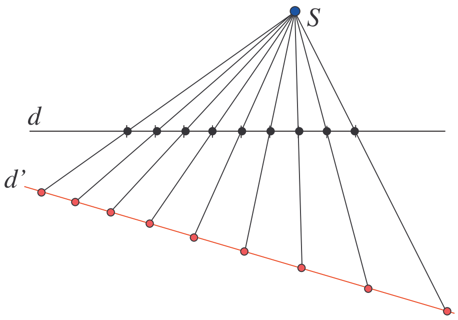

Figure 4: How would the other line get a regular graduation?

But at first we look at what happens if we take the point S far away, let's say at infinity. The lines through that point seem parallel as the sunbeams and regularity is preserved, even if the distances between marks is different.

Graduation from line to another line (link to JavaSketchpad animation)
`http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/Graduation.htm`

1.1 Thales's theorem

Plutarch recounts a story which, if accurate, would mean that Thales (624-546 B.C.) was getting close to the idea of similar triangles:
"... without trouble or the assistance of any instrument Thales merely set up a stick at the extremity of the shadow cast by the pyramid and, having thus made two triangles by the impact of the sun's rays, ... showed that the pyramid has to the stick the same ratio which the shadow of the pyramid has to the shadow of the stick."
We don't know if this story is true, but we may think that it is the reason why in France the following theorm is called Thales's theorem (in many european countries, Thales's theorem is the theorem that says that the points of a circle C are those who view a diameter of C under a straight angle).
The notion of algebraic measure is also specific to French tradition. It is sometime quite convenient and it is allways quite easy to use. In this lecture, we shall use the French terminology.

1.1.1 Elementary form of Thales's theorem

- *Problem 1.* Divide a given segment drawn on transparent paper in three segments of equal lengths, using equidistant parallel lines drawn on another sheet of paper.
- *Problem 2.* Construct with ruler and compass the point M of the segment AB such that

$$\text{dist}(A, M) = \frac{2}{5}\,\text{dist}(A, B).$$

An idea of a solution is seen in Figure 5.

Figure 5: Dividing a segment

Note that if we just want M to be on the straight line $\overleftrightarrow{AB}$ we have two solutions. We can distinguish the solutions using vectors:

$$\overrightarrow{AM_1} = \frac{2}{5}\overrightarrow{AB} \qquad \text{and} \qquad \overrightarrow{AM_2} = -\frac{2}{5}\overrightarrow{AB}$$

or with real line arithmetic notations:

$$M_1 - A = \frac{2}{5}(B - A) \qquad \text{and} \qquad M_2 - A = -\frac{2}{5}(B - A).$$

1.1.2 Thales theorem with 3 parallel lines

Given a couple of points (P, Q) on a (straight) line d of a usual plane $\mathcal{P}$, we can associate to it three quantities:

1. the distance between the two points, denoted by PQ or $\mathrm{dist}(P, Q)$. That supposes that a unit length has been chosen. Note that if we have three points P, Q and R, then the ratio $\frac{\mathrm{dist}(P,Q)}{\mathrm{dist}(P,R)}$ does not depend on the choice of the unit length;
2. the vector $\overrightarrow{PQ}$, also denoted by $Q - P$;
3. the *algebraic value* $\overline{PQ}$ which is a real number combining an absolute value

$$\left|\overline{PQ}\right| = \mathrm{dist}(P, Q),$$

and a sign. The sign of $\overline{PQ}$ depends on the orientation of the line d. If you choose a unit vector $\mathbf{u}$ to orient d, then $\overline{PQ}$ is defined by:

$$\overrightarrow{PQ} = \overline{PQ}\,\mathbf{u}$$

or with the other notation $Q - P = \overline{PQ}\,\mathbf{u}$. Note that the ratio $\frac{\overline{PQ}}{\overline{RS}}$ is independent of the choice of $\mathbf{u}$.

Theorem 1.1.1 (Thales's theorem) If two lines d and d' are intersected by three parallel lines in respectively A and A', B and B', C and C' (see Figure 6), then:

$$\frac{\overline{A'B'}}{\overline{A'C'}} = \frac{\overline{AB}}{\overline{AC}}.$$

Exercise 1.1.2 Criticize the formulation of the preceding theorem. Hint: *dividing by 0 has no meaning*.

A simple proof. Suppose $\frac{\overline{AB}}{\overline{AC}} = \frac{2}{5}$. Take the points E, M and N on d such that (see Figure 7)

$$\mathrm{dist}(A, E) = \mathrm{dist}(E, B) = \mathrm{dist}(B, M) = \mathrm{dist}(M, N) = \mathrm{dist}(N, C).$$

Draw the lines $\overleftrightarrow{EE'}$, $\overleftrightarrow{MM'}$ and $\overleftrightarrow{NN'}$ parallel to $\overleftrightarrow{AA'}$. Draw the lines $\overleftrightarrow{AE'_1}$, $\overleftrightarrow{EB'_1}$, $\overleftrightarrow{BM'_1}$, $\overleftrightarrow{MN'_1}$ and $\overleftrightarrow{NC'_1}$ parallel to d'.

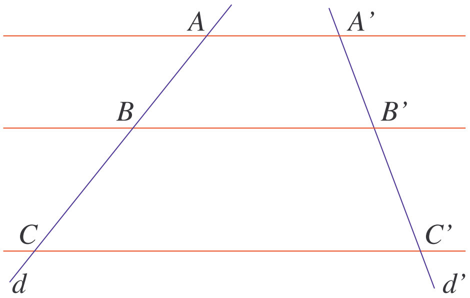

Figure 6: Three parallel lines

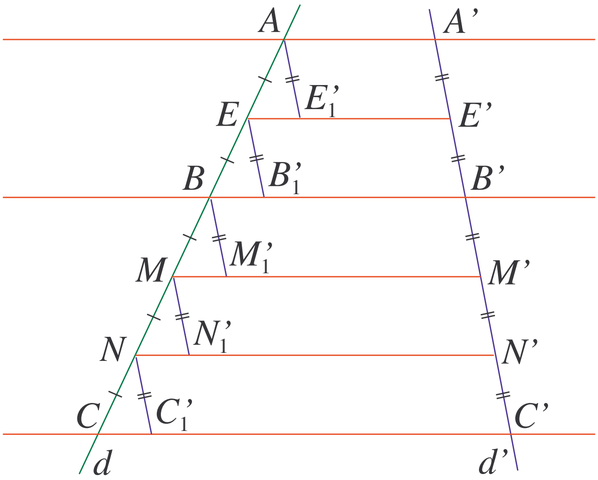

Figure 7: Three parallel lines, simple proof

Since $AA'E'E_1'$, $EE'B'B_1'$, … are parallelograms we have

$$\begin{aligned}&\text{dist}(A',E')=\text{dist}(A,E_1'),\ \text{dist}(E',B')=\text{dist}(E,B_1'),\ \text{dist}(B',M')=\text{dist}(B,M_1'),\\&\text{dist}(M',N')=\text{dist}(M,N_1'),\ \text{dist}(N',C')=\text{dist}(N,C_1').\end{aligned}$$

The triangles AEE_1', EBB_1', BMM_1', MNN_1' and NCC_1' are equal, that is isometric. Thus

$$\text{dist}(A,E_1')=\text{dist}(E,B_1')=\text{dist}(B,M_1')=\text{dist}(M,N_1')=\text{dist}(N,C_1'),$$

and so

$$\text{dist}(A',E')=\text{dist}(E',B')=\text{dist}(B',M')=\text{dist}(M',N')=\text{dist}(N',C').$$

Consequently $\frac{\text{dist}(A',B')}{\text{dist}(A',C')}=\frac{2}{5}$. We see on the figure that A', B' and C' are in the same order as A, B and C. Then $\frac{\overline{A'B'}}{\overline{A'C'}}$ has the same sign as $\frac{\overline{AB}}{\overline{AC}}$ and finally $\frac{\overline{A'B'}}{\overline{A'C'}}=\frac{2}{5}$. ■

Another proof of the theorem using linear algebra. We define the algebraic values on the three parallel lines with the same vector $\mathbf{u}$. Let $\mathbf{v}$ and $\mathbf{v}'$ be unit vectors of d and d'. Since the parallel lines intersect d, the vectors $\mathbf{u}$ and $\mathbf{v}$ are independent and they form a basis. Let us decompose $\mathbf{v}'$ on that basis: $\mathbf{v}'=\alpha\mathbf{u}+\beta\mathbf{v}$. Since the parallel lines intersect d', the vectors $\mathbf{u}$ and $\mathbf{v}'$ are independent, which imposes that $\beta\neq 0$.

The definition of the algebraic values gives us

$$\overrightarrow{A'B'}=\overline{A'B'}\,\mathbf{v}'=\alpha\overline{A'B'}\,\mathbf{u}+\beta\overline{A'B'}\,\mathbf{v}$$

and also

$$\overrightarrow{A'B'} = \overrightarrow{A'A} + \overrightarrow{AB} + \overrightarrow{BB'} = (\overline{A'A} + \overline{BB'})\,\mathbf{u} + \overline{AB}\,\mathbf{v}.$$

The unicity of the decomposition on a basis gives

$$\beta \overline{A'B'} = \overline{AB}.$$

In the same way we get

$$\overline{AC} = \beta \overline{A'C'}.$$

Multiplying member by member these equalities and dividing by $\beta \neq 0$ we finally get:

$$\overline{A'B'} \cdot \overline{AC} = \overline{A'C'} \cdot \overline{AB}.$$

■

Another way. If you don't like the proofs given above or if you don't know enough linear algebra, take this theorem as an axiom and …you have nothing to prove! To do mathematics, we need building blocks, or axioms. You can construct mathematical geometry as a part of linear algebra, but you can do geometry without linear algebra, which has been done for centuries. In fact, most of the basic ideas in linear algebra are coming from geometry. ■

Exercise 1.1.3 Let ABC be a triangle and M_0 a point of $\overleftrightarrow{BC}$. The parallel line to $\overleftrightarrow{AB}$ through M_0 cuts $\overleftrightarrow{CA}$ in a point M_1,

the parallel line to $\overleftrightarrow{BC}$ through M_1 cuts $\overleftrightarrow{AB}$ in a point M_2,

the parallel line to $\overleftrightarrow{CA}$ through M_2 cuts …and so on, see Figure 8.

Show that $M_6 = M_0$.

Is it possible that $M_3 = M_0$?

Is it true that $\overleftrightarrow{M_1M_2}$, $\overleftrightarrow{M_3M_4}$ and $\overleftrightarrow{M_5M_0}$ are concurrent?

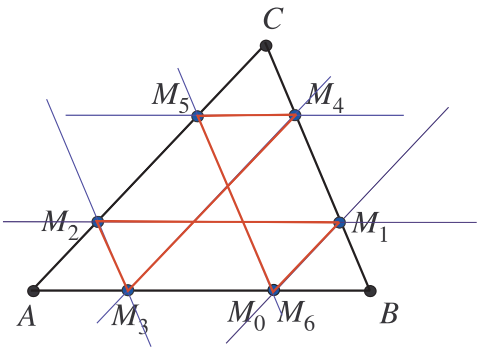

Figure 8: Triangle and parallel lines

See also further problems in

Successive lines parallel to the sides of a triangle (link to JavaSketchpad animation)
`http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/TriangleAndParallelLines.htm`

1.1.3 Thales's theorem: preservation of regularity

Definition 1.1.4 A *graduation* of a (straight) line d is a bijection g of d onto $\mathbb{R}$. The graduation is called *regular* if it is such that for any three points A, B and C of d with $A \neq C$ one has:

$$\frac{g(B)-g(A)}{g(C)-g(A)} = \frac{\overline{AB}}{\overline{AC}}.$$

Remark 1.1.5 The image $g(M)$ of a point M is often called the abscissa of M and denoted g_M or x_M.

Comment. We say that two lines have the same direction if they are parallel. Do not mix direction and orientation: a line has one direction and two possible orientations. How can we define the conceptof direction? One way is to notice that parallelism is an equivalence relation in the set of lines. One could define the direction of a line d as the equivalence class of d, that is the set of all lines parallel to d.

Another way to define the direction of a line d is to use vectors: given a line d we can construct the set of vectors $\{\overrightarrow{AB} \mid A \in d \text{ and } B \in d\}$. All the vectors belonging to that set are collinear. The set $\{\overrightarrow{AB} \mid A \in d \text{ and } B \in d\}$ is a one dimensional subspace of the space of all the vectors in the plane. A line d' is parallel to d if and only if the set $\{\overrightarrow{A'B'} \mid A' \in d' \text{ and } B' \in d'\}$ is the same as $\{\overrightarrow{AB} \mid A \in d \text{ and } B \in d\}$. This leads to the following definition.

Definition 1.1.6 Let $\mathcal{P}$ be a plane and $\overrightarrow{\mathcal{P}}$ the 2-dimensional vector space formed by all the vectors of $\mathcal{P}$. Let d be a line. We call *direction* of d the one-dimensional subspace of $\overrightarrow{\mathcal{P}}$

$$\delta := \{\mathbf{u} \in \overrightarrow{\mathcal{P}} \mid \exists\, A \in d \text{ and } \exists\, B \in d : \quad \mathbf{u} = \overrightarrow{AB}\}.$$

Proposition 1.1.7 Two lines are parallel if and only if they have same direction.

Proposition 1.1.8 Given a direction δ and two lines d and d' whose directions are both different from δ, for any point M of d there is a unique point M' belonging to d' and to the line through M with direction δ (see Figure 9).

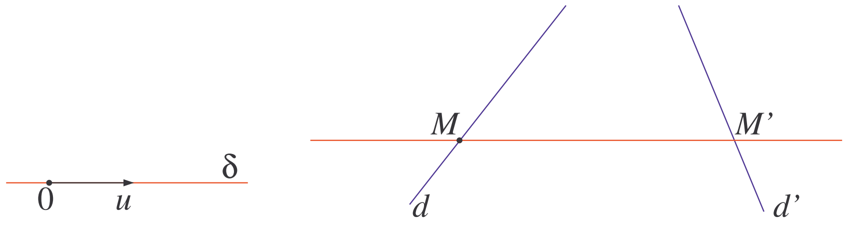

Figure 9: Direction induces a unique point on a line

Proof. Let $M \in d$, there is a unique line l going through M and with direction δ. Since l and d' do not have the same direction, they intersect in a unique point M'. ■

Definition 1.1.9 Let δ be a direction and let d and d' be two lines whose directions are both different from δ. Denote by $p_{\delta,d\to d'}$ the map of d onto d' that associates to any point M on d the unique point M' on d' such that the vector $\overrightarrow{MM'}$ belongs to δ (that is M' belongs to the line through M with direction δ).

That map $p_{\delta,d\to d'}$ is called *parallel projection* of d onto d' parallel to the direction δ:

$$p_{\delta,d\to d'} : d \longrightarrow d',\ M \mapsto M' \quad \text{such that} \quad \overrightarrow{MM'} \in \delta.$$

Notice that $p_{\delta,d\to d'}$ is a bijection. One has:

$$(p_{\delta,d\to d'})^{-1} = p_{\delta,d'\to d}$$

A way to interpret Thales's theorem is to say that parallel projections from a line onto an other preserve regular graduations:

Theorem 1.1.10 Let δ be a direction and d and d' two lines whose directions are both different from δ and let $p_{\delta,d'\to d}$ be the parallel projection from d' onto d parallel to the direction δ. For any regular graduation g of d, $g \circ p_{\delta,d'\to d}$ is a regular graduation of d'.

We could have a stronger statement: the only continuous graduations of lines in an affine real plane which are compatible with Thales's theorem are the regular ones. Exercise 1.1.3 gives a hint: the midpoints of a graduation compatible with Thales's theorem are those of a regular graduation.

1.1.4 Thales's theorem expressed with triangles

Thales is told to have used the theorem we have studied to measure the height of the Egyptian pyramids, comparing the shadows of the buildings and that of a stick. But to do so he needed the following version of the theorem:

Theorem 1.1.11 Let d and d' be two lines intersecting in a point A. Let B and C be two points of d different from A and let B' and C' be two points of d' different from A. If $\overleftrightarrow{BB'} \parallel \overleftrightarrow{CC'}$ then:

$$\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AB'}}{\overline{AC'}} = \frac{\overline{BB'}}{\overline{CC'}}.$$

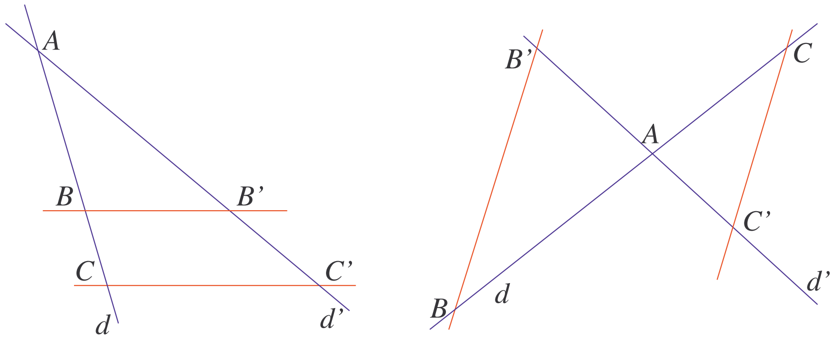

Figure 10: Thales and triangle ratios

Proof. We can draw a line through A parallel to $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ and therefore the first equality holds. Let us call λ the real number such that $\overline{AB} = \lambda\,\overline{AC}$ and $\overline{AB'} = \lambda\,\overline{AC'}$. We have:

$$\overrightarrow{BA} = \lambda\overrightarrow{CA} \quad \text{and} \quad \overrightarrow{AB'} = \lambda\overrightarrow{AC'}.$$

A straightforward computation gives then:

$$\overrightarrow{BB'} = \overrightarrow{BA} + \overrightarrow{AB'} = \lambda\overrightarrow{CA} + \lambda\overrightarrow{AC'} = \lambda\overrightarrow{CC'}.$$

■

1.1.5 Converse of Thales's theorem

Theorem 1.1.12 Let A, B and C be three collinear points and A', B' and C' be three collinear points such that the vectors $\overrightarrow{AA'}$ and $\overrightarrow{AC}$ are independent. If C' belongs to the parallel to $\overleftrightarrow{AA'}$ through C and if

$$\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{A'B'}}{\overline{A'C'}},$$

then B' belongs to the parallel to $\overleftrightarrow{AA'}$ through B.

Proof. Let d be the line through A, B and C. Let d' be the line through A', B' and C'. Consider the projection from d onto d' parallel to the direction of the line $\overleftrightarrow{AA'}$. This projection transforms A into A', C into C' and B into a point B_1.

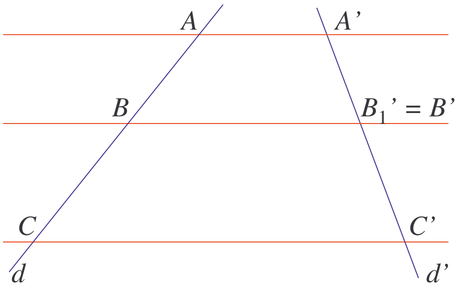

Figure 11: Converse of Thales's theorem

This point B_1 is the same as B' since they both belong to d' and:

$$\frac{\overline{A'B_1}}{\overline{A'C'}} = \frac{\overline{AB}}{\overline{AC}} = \frac{\overline{A'B'}}{\overline{A'C'}}.$$

■

Exercise 1.1.13 Let ABC be a triangle. Let A' be the midpoint of BC, B' be the midpoint of CA and C' be the midpoint of AB. The lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are called the *medians* of the triangle ABC. Show using Thales's theorem that the medians of a triangle are concurrent.

Exercise 1.1.14 Let ABC be a triangle, B' be the midpoint of CA and C' be the midpoint of AB. Let M be a point on $\overleftrightarrow{B'C'}$ different from B' and from C'. Let P be the intersection of $\overleftrightarrow{BM}$ and of the parallel to $\overleftrightarrow{AB}$ through B' and let Q be the intersection of $\overleftrightarrow{CM}$ and of the parallel to $\overleftrightarrow{AC}$ through C'.

1) Show that $\overleftrightarrow{PC}$ and $\overleftrightarrow{QB}$ are parallel.

2) Show that $\overleftrightarrow{AM}$ is parallel to $\overleftrightarrow{PC}$ and $\overleftrightarrow{QB}$.

3) Let A' be the midpoint of BC. Show that the parallel to $\overleftrightarrow{AB}$ through B' and the parallel to $\overleftrightarrow{AC}$ through C' pass through A'.

4) Let P_1 be the intersection of $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{QA}$. Show that

$$\frac{\overline{B'P_1}}{\overline{A'P_1}} = \frac{\overline{B'P}}{\overline{A'P}}.$$

5) Show that A, P and Q are collinear.

Exercise 1.1.15 Solve preceding exercise analytically: show that A, P and Q are collinear.

1.2 Concurrent lines and collinear points

1.2.1 Menelaus's Theorem

Theorem 1.2.1 Let ABC be a triangle, P a point belonging to $\overleftrightarrow{BC}$, Q a point belonging to $\overleftrightarrow{CA}$ and R a point belonging to $\overleftrightarrow{AB}$. The points P, Q and R are collinear if and only if:

$$\frac{\overline{PB}}{\overline{PC}} \cdot \frac{\overline{QC}}{\overline{QA}} \cdot \frac{\overline{RA}}{\overline{RB}} = 1.$$

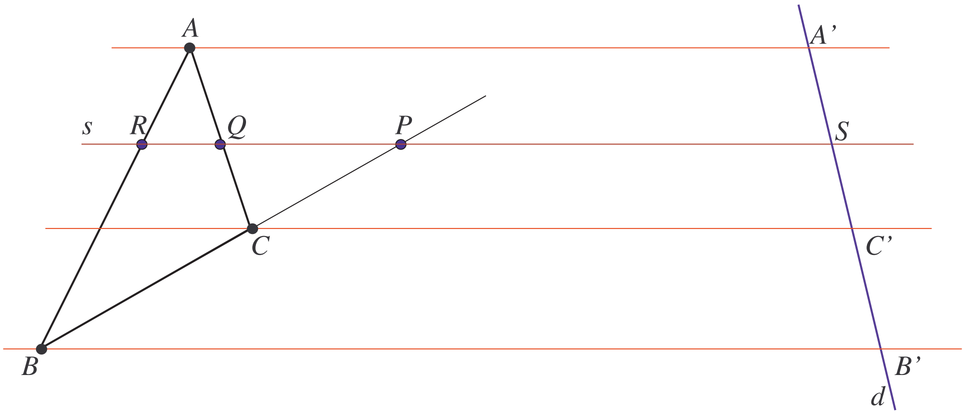

Figure 12: Menelaus's Theorem

Proof. 1) Suppose P, Q and R are collinear on a line we call s. Call σ the direction of s. Choose any line d whose direction is different from σ. Call S the intersection of d and s, A' the intersection of d and the parallel to s through A, B' the intersection of d and the parallel to s through B and C' the intersection of d and the parallel to s through C. Using Thales's theorem, we get:

$$\frac{\overline{SB'}}{\overline{SC'}} = \frac{\overline{PB}}{\overline{PC}}, \qquad \frac{\overline{SC'}}{\overline{SA'}} = \frac{\overline{QC}}{\overline{QA}} \qquad \text{and} \qquad \frac{\overline{SA'}}{\overline{SB'}} = \frac{\overline{RA}}{\overline{RB}}.$$

The relation we want to show is then equivalent to the following trivial one:

$$\frac{\overline{SB'}}{\overline{SC'}} \cdot \frac{\overline{SC'}}{\overline{SA'}} \cdot \frac{\overline{SA'}}{\overline{SB'}} = 1.$$

2) Conversely, you just have to notice that:

$$\frac{\overline{R_1A}}{\overline{R_1B}} = \frac{\overline{RA}}{\overline{RB}} \quad \Longrightarrow \quad R_1 = R.$$

■

Remark 1.2.2 You can get rid of some special cases where the denominators may be zero by writing the relation as:

$$\overline{PB} \cdot \overline{QC} \cdot \overline{RA} = \overline{PC} \cdot \overline{QA} \cdot \overline{RB}.$$

Remark 1.2.3 This shows also that in some cases algebraic measures can be more convenient than vectors.

Exercise 1.2.4 Show that the midpoints of the three diagonals of a quadrilateral are collinear.

Hint: Let ABC be a triangle and let D, E and F be three collinear points belonging respectively to $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$. The four lines through point triples BCD, CAE, ABF and DEF are called the *sides* of the quadrilateral. The vertices of that quadrilateral are the six points A, B, C, D, E and F. The diagonals are AD, BE and CF. Let us call the midpoints of these segments respectively P, Q and R. We have to show that P, Q and R are collinear. Introduce the points: I the intersection of $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$, J the intersection of $\overleftrightarrow{CF}$ and $\overleftrightarrow{AD}$ and K the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$.

Express the Menelaus relation $(*)$ for the three points P, Q and R relatively to the triangle IJK. Notice that:

$$\frac{\overline{PJ}}{\overline{PK}} = \frac{\overline{AJ} + \overline{DJ}}{\overline{AK} + \overline{DK}}$$

Show that $(*)$ is equivalent to:

$$\begin{aligned} & \overline{AJ} \cdot \overline{BK} \cdot \overline{CI} + \overline{AJ} \cdot \overline{EK} \cdot \overline{FI} + \overline{DJ} \cdot \overline{BK} \cdot \overline{FI} + \overline{DJ} \cdot \overline{EK} \cdot \overline{CI} \\ = \; & \overline{AK} \cdot \overline{BI} \cdot \overline{CJ} + \overline{AK} \cdot \overline{EI} \cdot \overline{FJ} + \overline{DK} \cdot \overline{BI} \cdot \overline{FJ} + \overline{DK} \cdot \overline{EI} \cdot \overline{CJ} \end{aligned}$$

Show that $(*)$ is equivalent to:

$$\begin{aligned} & \overline{CF}(\overline{AJ} \cdot \overline{BK} - \overline{AJ} \cdot \overline{EK} - \overline{DJ} \cdot \overline{EK} + \overline{AJ} \cdot \overline{BK}) \\ = \; & \overline{CF}(\overline{AK} \cdot \overline{BI} - \overline{AK} \cdot \overline{EI} - \overline{DK} \cdot \overline{BI} + \overline{DK} \cdot \overline{EI}). \end{aligned}$$

Exercise 1.2.5 Let $ABCD$ be a tetrahedron. A plane Π cuts the edges $\overleftrightarrow{AB}$ in P, $\overleftrightarrow{BC}$ in Q, $\overleftrightarrow{CD}$ in R and $\overleftrightarrow{DA}$ in S. Show that

$$\frac{\overline{PA}}{\overline{PB}} \cdot \frac{\overline{QB}}{\overline{QC}} \cdot \frac{\overline{RC}}{\overline{RD}} \cdot \frac{\overline{SD}}{\overline{SA}} = 1.$$

1.2.2 Cross ratio

Recall that the *ratio* of three collinear points A, B and C is

$$[A; B, C] := \frac{\overline{AB}}{\overline{AC}}.$$

It is a quantity which is conserved by parallel projection but not conserved by central projection. The quantity conserved by central projection is the cross-ratio, involving 4 points.

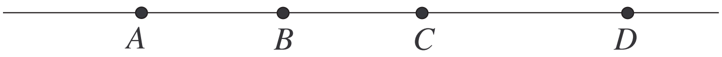

Figure 13: The four points

Definition 1.2.6 Let A, B, C and D be four distinct collinear points. The *cross-ratio* of these four points in that order (see Figure 13), is the real number

$$[A, B; C, D] := \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}}.$$

If we use a regular graduation of the line mapping each point M on its abscissa x_M, we can write:

$$[A, B; C, D] = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}} = \frac{(x_C - x_A)(x_D - x_B)}{(x_D - x_A)(x_C - x_B)}.$$

The last expression, denoted by $[x_A, x_B; x_C, x_D]$, is called the cross-ratio of the four numbers x_A, x_B, x_C and x_D.

Exercise 1.2.7 Let $\chi := [A, B; C, D]$.

1) Compute in terms of χ the cross-ratios $[B, A; D, C]$, $[C, D; A, B]$ and $[D, C; B, A]$.

2) Show that $[A, B; D, C] = \frac{1}{\chi}$ and $[A, C; B, D] = 1 - \chi$.

3) Compute in terms of χ the cross-ratios $[A, C; D, B]$, $[A, D; B, C]$ and $[A, D; C, B]$.

4) Compute all the cross-ratios you can build with the four points A, B, C and D. In what case do you find less than 6 distinct values?

Exercise 1.2.8 Let a, b, c and d be complex numbers, affixes of points A, B, C and D. Show that their cross ratio $\chi := [a, b; c, d]$ is a real number if and only if the four points are on a circle or a line.

Theorem 1.2.9 Let S be a point and let a, b, c and d be four distinct lines concurrent in S. Let δ and δ' be two lines intersecting a, b, c and d, respectively, in A, B, C and D and in A', B', C' and D'. Then

$$[A, B; C, D] = [A', B'; C', D'].$$

Proof. See Figure 14: Draw the line through C parallel to d. It intersects a in a point A_1 and b in a point B_1. Using Thales's theorem (third form), we get

$$[A, B; C, D] = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{BD}}{\overline{BC}} = \frac{\overline{A_1C}}{\overline{SD}} \cdot \frac{\overline{SD}}{\overline{B_1C}} = \frac{\overline{A_1C}}{\overline{B_1C}}.$$

Drawing the parallel to d through C', intersecting a in a point A'_1 and b in B'_1, we get

$$[A', B'; C', D'] = \frac{\overline{A'_1C'}}{\overline{B'_1C'}}$$

and we conclude by using Thales's theorem once more. ■

Remark 1.2.10 This theorem shows that we can define the cross-ratio of four concurrent lines, which is called a *bundle* of four concurrent lines.

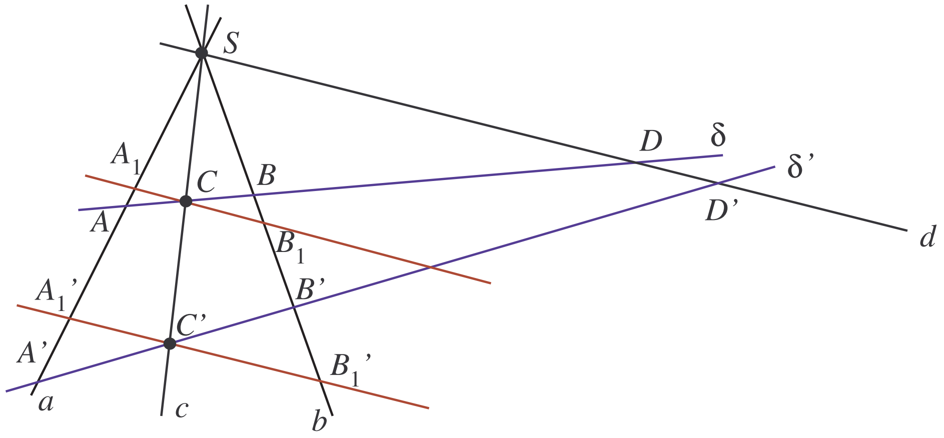

Figure 14: Cross ratio of lines

Definition 1.2.11 Let a, b, c and d be four concurrent lines or four parallel lines. The *cross-ratio* $[a, b; c, d]$ of these *four lines* in that order is the number $[A, B; C, D]$, where A, B, C and D are intersections of a, b, c and d with any line δ:

$$[a, b; c, d] := [A, B; C, D]$$

Exercise 1.2.12 Let A, B, C and D be four fixed points on a conic Γ. Let a point M describe the conic.

1) Suppose Γ is a circle. Show that $[\overleftrightarrow{MA}, \overleftrightarrow{MD}; \overleftrightarrow{MB}, \overleftrightarrow{MC}]$ is independent of the choice of M on $\Gamma \setminus \{A, B, C, D\}$.

2) Generalize to any conic.

1.2.3 Harmonic set of points, harmonic bundle of lines

Harmonicity is a way to generalize the idea of midpoint: in Figure 14, A, B, C and D form a harmonic set of points iff C is the midpoint of A_1B_1. The typical harmonic set is $(A, B; M, \infty)$ where M is the midpoint of segment $[AB]$ and ∞ is the point at infinity.

Definition 1.2.13 Two pairs of collinear points $\{A, B\}$ and $\{C, D\}$ are said to be *harmonic* if $[A, B; C, D] = -1$. We say also in that case that A, B, C and D is a *harmonic set* of points. Four lines a, b, c and d of a bundle are said to be *harmonic* if $[a, b; c, d] = -1$.

Theorem 1.2.14 In a quadrilateral the three diagonal points and the six vertices form three harmonic sets.

More explicitly (see Figure 15): let A, B, C, D, E and F be the vertices of a quadrilateral with B, C and D collinear, C, A and E collinear, A, B and F collinear and D, E and F collinear. We call I the intersection of $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$, J the intersection of $\overleftrightarrow{CF}$ and $\overleftrightarrow{AD}$, K the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$. Then

$$[A, D; J, K] = -1, \quad [B, E; K, I] = -1 \quad \text{and} \quad [C, F; I, J] = -1.$$

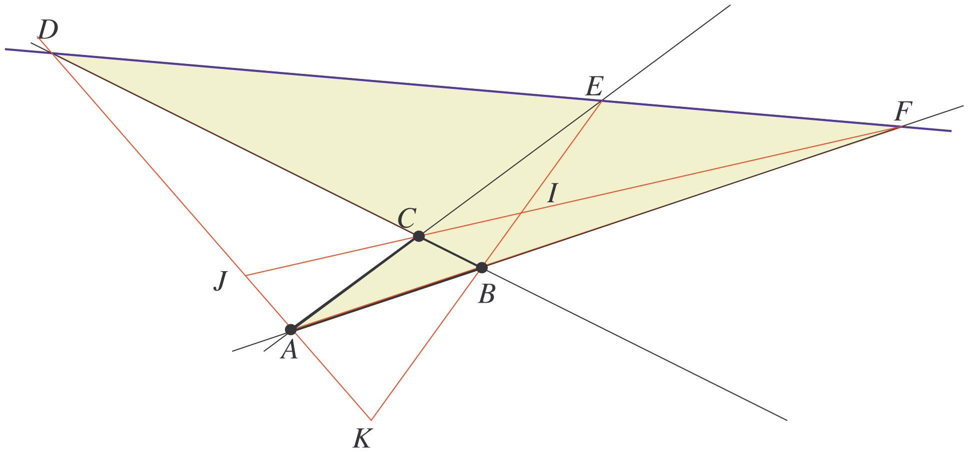

Figure 15: Quadrilateral with harmonic sets $ADJK$, $BEKI$ and $CFIJ$

Proof. Let us introduce the cross ratios of bundles of lines:

$$\begin{aligned}[A, D; J, K] &= [\overleftrightarrow{EA}, \overleftrightarrow{ED}; \overleftrightarrow{EJ}, \overleftrightarrow{EK}] = [C, F; J, I] \\ &= [\overleftrightarrow{BC}, \overleftrightarrow{BF}; \overleftrightarrow{BJ}, \overleftrightarrow{BI}] = [D, A; J, K] = \frac{1}{[A, D; J, K]}.\end{aligned}$$

Thus $[A, D; J, K]^2 = 1$. If the quadrilateral is a real one, i.e. if the four sides are distinct lines, then $[A, D; J, K] \neq 1$ and so $[A, D; J, K] = -1$. ■

Exercise 1.2.15 With the above notations, show that

$$[\overleftrightarrow{IA}, \overleftrightarrow{ID}; \overleftrightarrow{IJ}, \overleftrightarrow{IK}] = -1$$

Exercise 1.2.16 Let P, Q and R be three collinear points on a line δ. Choose any two points T and U collinear with R. Call V the intersection of $\overleftrightarrow{PT}$ and $\overleftrightarrow{QU}$, W the intersection of $\overleftrightarrow{PU}$ and $\overleftrightarrow{QT}$ and S the intersection of $\overleftrightarrow{VW}$ with δ. Show that S is independent of the choices of T and U.

Exercise 1.2.17 Draw a line between two points with a ruler which is too short.

The notion of harmonic division of a segment has its origin in music: to make a perfect accord, you have to divide a string in points of abscissae $0, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}$, see Figure 16.

Figure 16: Perfect accords

Exercise 1.2.18 Let ABC be a triangle, A' the midpoint of BC and δ the parallel line to $\overleftrightarrow{BC}$ through A (see Figure 17).

Show that $[\overleftrightarrow{AA'}, \delta; \overleftrightarrow{AB}, \overleftrightarrow{AC}] = -1$.

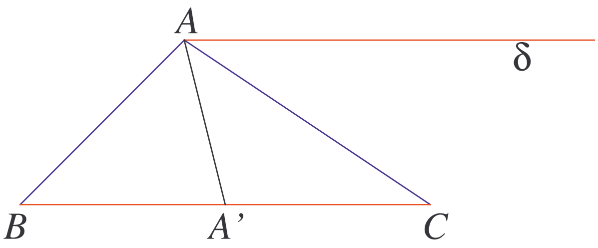

Figure 17: Figure for Exercise 1.2.18

Exercise 1.2.19 Let d and d' be two secant lines and let δ and δ' be the bisectors of the angles made by d and d' (see Figure 18).

Show: $[d, d'; \delta, \delta'] = -1$.

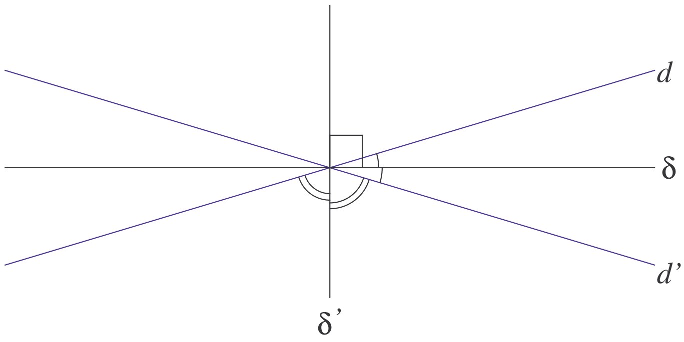

Figure 18: Figure for Exercise 1.2.19

1.2.4 Ceva's theorem

Theorem 1.2.20 Let ABC be a triangle, P a point belonging to $\overleftrightarrow{BC}$, Q a point belonging to $\overleftrightarrow{CA}$ and R a point belonging to $\overleftrightarrow{AB}$ (see Figure 19). The lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CR}$ are concurrent if and only if:

$$\frac{\overline{PB}}{\overline{PC}} \cdot \frac{\overline{QC}}{\overline{QA}} \cdot \frac{\overline{RA}}{\overline{RB}} = -1.$$

Proof. Let P' be the intersection of $\overleftrightarrow{QR}$ and $\overleftrightarrow{BC}$, let G be the intersection of $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CR}$ and let P_1 be the intersection of $\overleftrightarrow{AG}$ and $\overleftrightarrow{BC}$. The set of points B, C, P' and P_1 is harmonic, so:

$$\frac{\overline{P'B}}{\overline{P'C}} = -\frac{\overline{P_1B}}{\overline{P_1C}}.$$

Since Q, R and P' are collinear, Menelaus's theorem gives us:

$$\frac{\overline{P'B}}{\overline{P'C}} \cdot \frac{\overline{QC}}{\overline{QA}} \cdot \frac{\overline{RA}}{\overline{RB}} = 1.$$

We conclude by noticing that$\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CR}$ are concurrent if and only if $P = P_1$. ■

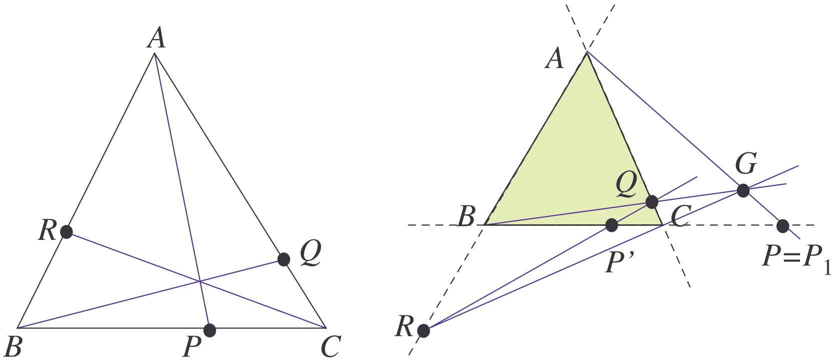

Figure 19: Concurrent lines of Ceva's Theorem

Exercise 1.2.21 Use Ceva's theorem to show that the medians of a triangle are concurrent.

Exercise 1.2.22 Use Ceva's theorem to show that the internal bisectors of a triangle are concurrent and that one internal and two external bisectors of a triangle are concurrent.

Exercise 1.2.23 Let ABC be a triangle in a plane Π, P a point belonging to $\overleftrightarrow{BC}$, Q a point belonging to $\overleftrightarrow{CA}$ and R a point belonging to $\overleftrightarrow{AB}$ such that the lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BQ}$ and $\overleftrightarrow{CR}$ are concurrent in a point G. Let S be a point in space not belonging to Π. Show that there is a plane Π' intersecting $\overleftrightarrow{SA}$, $\overleftrightarrow{SB}$, $\overleftrightarrow{SC}$ and $\overleftrightarrow{SG}$ in points A', B', C' and G' such that G' is the centroid of the triangle $A'B'C'$.

2 Orthocentric quadrangles

An equilateral triangle has one center. But for a triangle with sides of unequal lengths several centers are available, some of them might be outside the triangle. If you adjoin the orthocenter to the vertices of a triangle, you get an orthocentric quadrangle: four points such that any of them is the orthocenter of the three others. The Euler circle of a triangle is then the Euler circle of the orthocentric quadrangle.

2.1 Centers of a triangle

2.1.1 Barycentric coordinates

There are two possible definitions of barycentric coordinates in a plane relative to the three vertices of a triangle ABC. The first one, which we will call normalized barycentric coordinates, is conceptually easier but less practical. The second one, non-normalized barycentric coordinates, seems awkward in the beginning, but is much more satisfactory, intellectually and practically.

Definition 2.1.1 Let A, B and C be three non-collinear points of a real affine plane $\mathcal{P}$. The *normalized barycentric coordinates* of a point M (relative to A, B and C in that order) is the unique triplet (α, β, γ) such that for any point Ω:

$$M - \Omega = \alpha(A - \Omega) + \beta(B - \Omega) + \gamma(C - \Omega).$$

As a consequence $\alpha + \beta + \gamma = 1$ and we may write that relation

$$M = \alpha A + \beta B + \gamma C$$

If $\gamma = 0$, $\alpha + \beta = 1$ and M is the barycenter of A and B with weights α and β and we have

$$M = \alpha A + \beta B.$$

Remark 2.1.2 If we use coordinates, the relation translating that (α, β, γ) are the normalized barycentric coordinates of a point M relatively to A, B and C

$$M = \alpha A + \beta B + \gamma C$$

is just

$$\left\{ \begin{array}{lllllll} x_M & = & \alpha x_A & + & \beta x_B & + & \gamma x_C \\ y_M & = & \alpha y_A & + & \beta y_B & + & \gamma y_C \end{array} \right.$$

Theorem 2.1.3 Let $\mathcal{P}$ be a real affine plane, let $\overrightarrow{\mathcal{P}}$ be the vector space associated with $\mathcal{P}$, that is, the set of all the vectors of $\mathcal{P}$. The set

$$\mathcal{B} := (\mathbb{R}^* \times \mathcal{P}) \cup \overrightarrow{\mathcal{P}}$$

has the structure of a linear space of dimension 3 (see Figure 20), if we define the addition $+$ in $\mathcal{B}$ and the multiplication of an element of $\mathcal{B}$ by a real number in the following manner:

for any (α, A) and (β, B) belonging to $\mathbb{R}^* \times \mathcal{P}$, any $\mathbf{u}, \mathbf{v} \in \overrightarrow{\mathcal{P}}$ and any $\lambda \in \mathbb{C}$:

$$\left\{\begin{array}{rcl} (\alpha, A) + (\beta, B) & = & \left\{\begin{array}{ll} (\alpha+\beta, \frac{\alpha}{\alpha+\beta}A + \frac{\beta}{\alpha+\beta}B) & \text{if} \quad \alpha+\beta \neq 0 \\ \alpha\overrightarrow{BA} & \text{if} \quad \alpha+\beta = 0 \end{array}\right. \\ (\alpha, A) + \mathbf{u} & = & \mathbf{u} + (\alpha, A) = (\alpha, A + \frac{1}{\alpha}\mathbf{u}) \\ \mathbf{u} + \mathbf{v} \ \text{(computed in } \mathcal{B}) & = & \mathbf{u} + \mathbf{v} \ \text{(computed in } \overrightarrow{\mathcal{P}}) \end{array}\right.$$

$$\left\{\begin{array}{rcl} \lambda(\alpha, A) & = & \left\{\begin{array}{ll} (\lambda\alpha, A), & \text{if} \quad \lambda \neq 0 \\ 0, & \text{if} \quad \lambda = 0 \end{array}\right. \\ \lambda\mathbf{u} \ \text{(computed in } \mathcal{B}) & = & \lambda\mathbf{u} \ \text{(computed in } \overrightarrow{\mathcal{P}}) \end{array}\right.$$

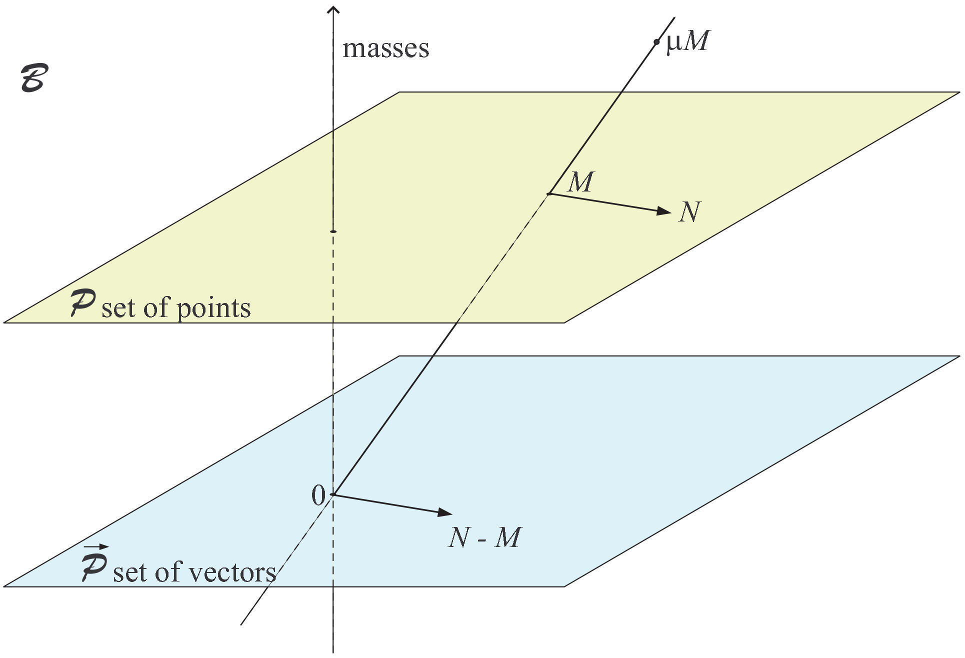

Figure 20: The space $\mathcal{B}$

Proof. Just check it! ■

Proposition 2.1.4 If three points A, B and C of $\mathcal{P}$ are non-collinear then $(1, A)$, $(1, B)$ and $(1, C)$ form a basis of $\mathcal{B}$.

Proof. We have to show that $(1, A)$, $(1, B)$ and $(1, C)$ generate $\mathcal{B}$ and that they are linearly independent.

1) (i) let $\mathbf{u}$ be a vector of the plane. Since A, B and C are non-collinear, $B - A$ and $C - A$ form a basis of $\overrightarrow{\mathcal{P}}$ and we can decompose $\mathbf{u}$ on that basis:

$$\mathbf{u} = \beta(B - A) + \gamma(C - A) = (\beta B + (-\beta)A) + (\gamma C + (-\gamma)A)$$

Using the rules in $\mathcal{B}$ given above, we get:

$$\begin{array}{rcl} \mathbf{u} & = & ((\beta, B) + (-\beta, A)) + ((\gamma, C) + (-\gamma, A)) \\ & = & (-\beta - \gamma, A) + (\beta, B) + (\gamma, C) \\ & = & (-\beta - \gamma)(1, A) + \beta(1, B) + \gamma(1, C). \end{array}$$

(ii) let (μ, M) be an element of $\mathbb{R}^* \times \mathcal{P}$. There exist three numbers α, β and γ such that $\alpha + \beta + \gamma = 1$ and $M = \alpha A + \beta B + \gamma C$. We can always suppose $\alpha + \beta \neq 0$. If not, choose α and γ or β and γ. Using the rules above we then get:

$$\begin{array}{rl} & \mu\alpha(1, A) + \mu\beta(1, B) + \mu\gamma(1, C) \\ = & (\mu\alpha, A) + (\mu\beta, B) + (\mu\gamma, C) \\ = & \left(\mu\alpha + \mu\beta, \dfrac{\alpha}{\alpha+\beta}A + \dfrac{\beta}{\alpha+\beta}B\right) + (\mu\gamma, C) \\ = & \left(\mu\alpha + \mu\beta + \mu\gamma, \dfrac{\mu\alpha+\mu\beta}{\mu\alpha+\mu\beta+\mu\gamma}\left[\dfrac{\alpha}{\alpha+\beta}A + \dfrac{\beta}{\alpha+\beta}B\right] + \dfrac{\mu\gamma}{\mu\alpha+\mu\beta+\mu\gamma}C\right) \\ = & (\mu, \alpha A + \beta B + \gamma C) = (\mu, M). \end{array}$$

2) Let us consider a linear combination of $(1, A)$, $(1, B)$ and $(1, C)$ which is 0:

$$\alpha(1, A) + \beta(1, B) + \gamma(1, C) = 0$$

and let us show that $\alpha = \beta = \gamma = 0$.

(i) If $\alpha + \beta + \gamma \neq 0$, we will get by the same computation as above

$$\alpha(1, A) + \beta(1, B) + \gamma(1, C) = (\alpha + \beta + \gamma, M)$$

which belongs to $\mathbb{R}^* \times \mathcal{P}$ and thus is not 0. So we must have $\alpha + \beta + \gamma = 0$.

(ii) Since $\alpha + \beta + \gamma = 0$, we can write $\gamma = -\alpha - \beta$ and

$$\begin{array}{rcl} 0 & = & \alpha(1, A) + \beta(1, B) + \gamma(1, C) \\ & = & (\alpha, A) + (\beta, B) + (-\alpha, C) + (-\beta, C) \\ & = & \alpha(A - C) + \beta(B - C), \end{array}$$

and since A, B and C are not collinear, $\alpha = \beta = 0$, and so $\gamma = -\alpha - \beta = 0$.

■

Notation 2.1.5 Instead of (α, A) we simply write αA and we write A instead of $(1, A)$. So A, B, C is a basis of $\mathcal{B}$.

Definition 2.1.6 The linear space $\mathcal{B}$, often denoted $\widetilde{\mathcal{P}}$, is called the *universal covering space* of $\mathcal{P}$: the elements of $\mathbb{R}^* \times \mathcal{P}$ are called *massive points*. If a massive point μM has coordinates (α, β, γ) in the basis (A, B, C), then we say that (α, β, γ) are *(non-normalized) barycentric coordinates* of M relatively to ABC:

$$\mu M = \alpha A + \beta B + \gamma C.$$

Remark 2.1.7 If (α, β, γ) are (non normalized) barycentric coordinates of M relatively to ABC, then $\alpha + \beta + \gamma \neq 0$. The number $\alpha + \beta + \gamma$ is the number μ of the relation

$$\mu M = \alpha A + \beta B + \gamma C.$$

If we have $\alpha + \beta + \gamma = 0$, then $\alpha A + \beta B + \gamma C$ is not a massive point, but a vector.

Remark 2.1.8 If (α, β, γ) are (non normalized) barycentric coordinates of M relatively to ABC, then the normalized barycentric coordinates of M are

$$\left(\frac{\alpha}{\alpha+\beta+\gamma}, \frac{\beta}{\alpha+\beta+\gamma}, \frac{\gamma}{\alpha+\beta+\gamma}\right).$$

Proposition 2.1.9 Let ABC be a triangle of $\mathcal{P}$, and let R, S and T be three points of $\mathcal{P}$ with (non normalized) barycentric coordinates relatively to ABC – respectively – $(\alpha_R, \beta_R, \gamma_R)$, $(\alpha_S, \beta_S, \gamma_S)$ and $(\alpha_T, \beta_T, \gamma_T)$. The points R, S and T are collinear if and only if

$$\begin{vmatrix} \alpha_R & \alpha_S & \alpha_T \\ \beta_R & \beta_S & \beta_T \\ \gamma_R & \gamma_S & \gamma_T \end{vmatrix} = 0.$$

Proof. Let $\rho = \alpha_R + \beta_R + \gamma_R$, $\sigma = \alpha_S + \beta_S + \gamma_S$ and $\tau = \alpha_T + \beta_T + \gamma_T$. The points R, S and T are collinear if and only if the three massive points ρR, σS and τT and the 0 of $\mathcal{B}$ are in one plane, which is equivalent to say that ρR, σS and τT are linearly dependent vectors (see Figure 21), or that the determinant of their components in a basis of $\mathcal{B}$ is equal to 0. ■

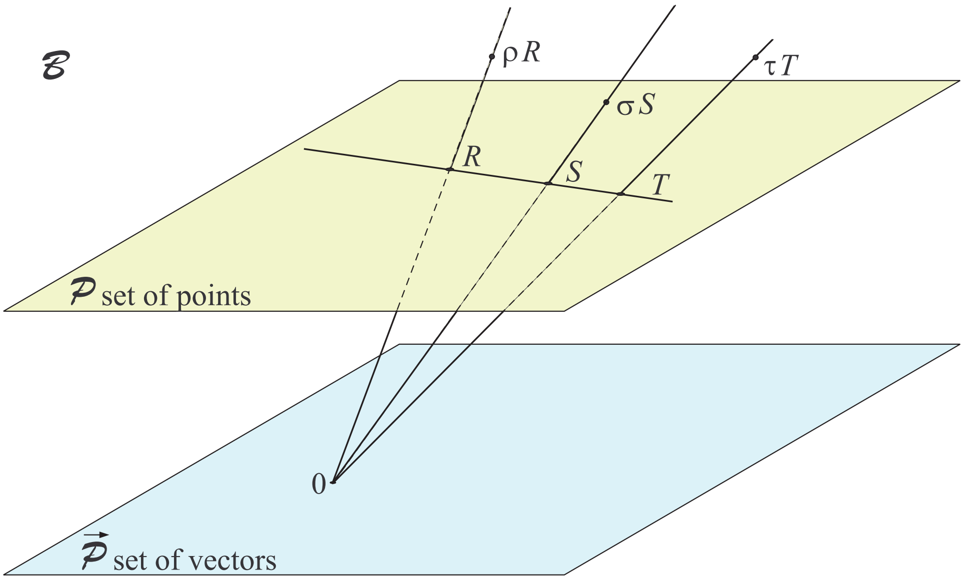

Figure 21: Triangle in $\mathcal{P}$

Corollary 2.1.10 Three points R, S and T with rectangular coordinates (x_R, y_R), (x_S, y_S) and (x_T, y_T) are collinear if and only if:

$$\begin{vmatrix} x_R & x_S & x_T \\ y_R & y_S & y_T \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Proof. Let I be the point with rectangular coordinates $(0, 1)$, J be the point with rectangular coordinates $(1, 0)$ and O be the point with rectangular coordinates $(0, 0)$. Then for any point M with rectangular coordinates (x_M, y_M) we have:

$$M - O = x_M(I - O) + y_M(J - O)$$

and so

$$M = x_M I + y_M J + (1 - x_M - y_M)O.$$

R, S and T are then collinear if and only if

$$\begin{vmatrix} x_R & x_S & x_T \\ y_R & y_S & y_T \\ 1 - x_R - y_R & 1 - x_S - y_S & 1 - x_T - y_T \end{vmatrix} = 0$$

We see that this relation is equivalent to the one we want by adding the first two lines to the third. ■

Remark 2.1.11 We get the result directly if we consider the points of $\mathcal{P}$ as being vectors in $\mathcal{B}$ expressed in the basis $(I - O, J - O, O)$. In fact, we then have:

$$M = x_M(I - O) + y_M(J - O) + 1O.$$

Exercise 2.1.12 Show that the general equation of a line in (non-normalized) barycentric coordinates is $a\alpha_M + b\beta_M + c\gamma_M = 0$, with $(a, b, c) \neq (0, 0, 0)$. Show that three lines with equations:

$$\left\{ \begin{array}{ccccccc} a_1\alpha & + & b_1\beta & + & c_1\gamma & = & 0 \\ a_2\alpha & + & b_2\beta & + & c_2\gamma & = & 0 \\ a_3\alpha & + & b_3\beta & + & c_3\gamma & = & 0 \end{array} \right.$$

are concurrent if and only if:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Exercise 2.1.13 a) Let ABC be a triangle. Show that if a point P of $\overleftrightarrow{BC}$ is such that

$$\lambda := \frac{\overline{PB}}{\overline{PC}} \neq 1,$$

then $(0, 1, -\lambda)$ are the barycentric coordinates of P relatively to the triangle ABC.

Show that the equation of the line $\overleftrightarrow{AP}$ is $0\alpha + \lambda\beta + 1\gamma = 0$. Show Menelaus's theorem and Cevas theorem using barycentric coordinates.

b) Show that the midpoints of the diagonals of a quadrilateral are collinear.

2.1.2 Centroid

Definition 2.1.14 The *centroid* of a triangle ABC is the point G such that

$$3G = A + B + C.$$

Definition 2.1.15 A *median* of a triangle is a line joining a vertex to the midpoint of the opposite side. If ABC is a triangle, we define the midpoints A', B' and C' to be the averages:

$$A' := \frac{1}{2}(B + C), \quad B' := \frac{1}{2}(C + A), \quad C' := \frac{1}{2}(A + B).$$

The medians of ABC are then $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$.

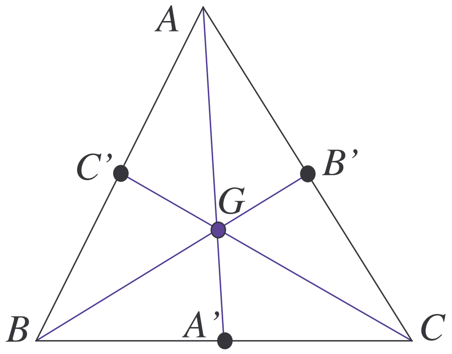

Figure 22: Medians and centroid

Theorem 2.1.16 The medians of a triangle are concurrent in the centroid.

Proof. See Figure 22. We have $3G = A + B + C = A + (B + C) = A + 2A'$. This relation means that G belongs to $\overleftrightarrow{AA'}$. ■

Exercise 2.1.17 1) Show that the diagonals of a parallelogram $RSTU$ cut each other in their midpoints (even if the parallelogram is degenerate); that means show that

$$S - R = T - U \quad \Longrightarrow \quad \frac{1}{2}(R + T) = \frac{1}{2}(S + U).$$

2) Let ABC be a triangle, B' the midpoint of CA and C' the midpoint of AB. Let G be the intersection of BB' and CC'. Let B$_2$ be the midpoint of BG and C_2 the midpoint of CG. Show that $B_2C_2B'C'$ is a parallelogram (see Figure 22).

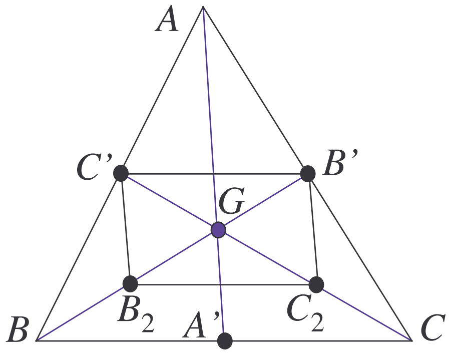

Figure 23: Midpoints and parallelogram

3) Give a proof of the fact that medians of a triangle are concurrent using both preceding results.

Exercise 2.1.18 Let G be the centroid of a triangle ABC in an euclidean plane and M any point of the plane. Show that $\text{dist}(M, A)^2 + \text{dist}(M, B)^2 + \text{dist}(M, C)^2$ is minimum when $M = G$.

2.1.3 Center of the circumscribed circle to a triangle

Proposition 2.1.19 Given three non collinear points A, B and C of an euclidean plane, there is a unique circle through these three points.

Proof. $A = B$ is impossible since the points are not collinear. The bisector line of AB is then well defined as well as the bisector line of AC. They can not be parallel since they are orthogonal to intersecting lines. Thus they have one and only one common point O.

Unicity: A circle through A, B and C must then have its center in O and going through A, it is unique.

Existence: Since O belongs to the bisector of AB, we have $OA = OB$, and similarly $OA = OC$ and the circle with center O and through A will pass also through B and C. ■

Definition 2.1.20 Let ABC be a triangle. The unique point O such that: $\text{dist}(O, A) = \text{dist}(O, B) = \text{dist}(O, C)$ is called the *center of the circum-circle* of the triangle ABC (see Figure 24).

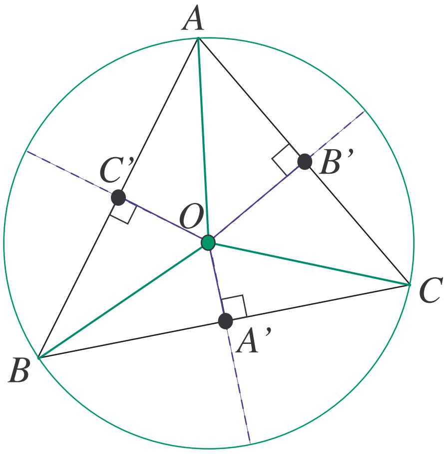

Figure 24: Center of the circum-circle

Exercise 2.1.21 Where is the circum-center of a right-angled triangle?

2.1.4 Orthocenter of a triangle

Definition 2.1.22 Let ABC be a triangle. The line through A orthogonal to $\overleftrightarrow{BC}$ is called an *altitude line* of ABC.

Proposition 2.1.23 The three altitude lines of a triangle are concurrent (see Figure 25).

G. Choquet wrote that progress in mathematics had been considerably delayed because geometers saw triangles, when they should have seen half-parallelograms. Compare for instance the formula giving the area of a triangle to that giving the area of a parallelogram. But given a triangle ABC, there are three parallelograms "naturally" associated to ABC: let A_1, B_1 and C_1 be such that $A_1 = -A + B + C$, $B_1 = A - B + C$ and $C_1 = A + B - C$. We have three parallelograms: ABA_1C, BCB_1A and CAC_1B.

Proof. Look at Figure 26. Think of the altitude lines of ABC as lines related to $A_1B_1C_1$. ■

Definition 2.1.24 The point common to the three altitudes of a triangle is called the *orthocenter* of the triangle.

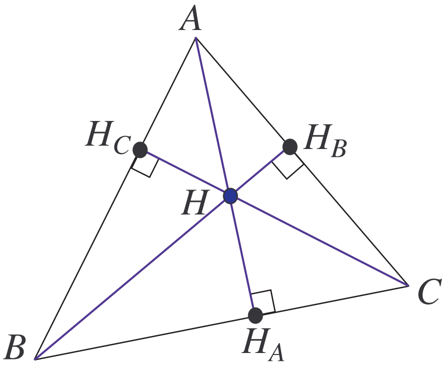

Figure 25: Altitude lines of a triangle

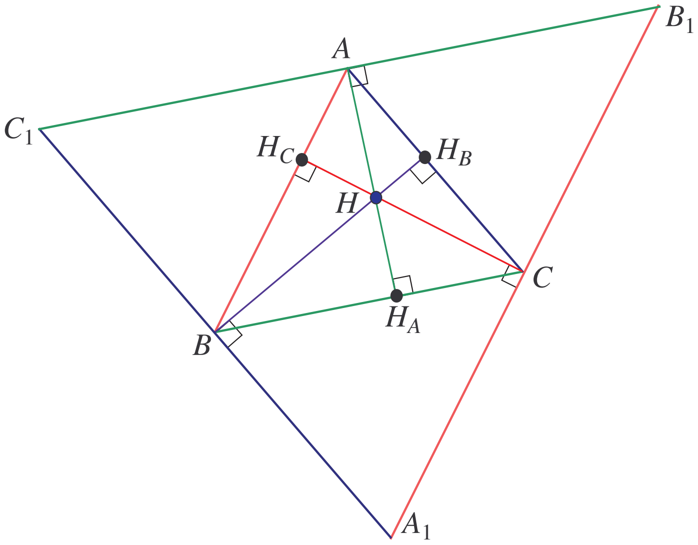

Figure 26: Three parallelograms of a triangle

Theorem 2.1.25 The centroid G, the center of the circum-circle O and the orthocenter H of a triangle are collinear (on a unique line called the *Euler line* of the triangle, if the triangle is not equilateral) and: $3G = H + 2O$.

Proof. Consider the central dilatation (or homothety) of center G and with factor $\kappa := -2$, which transforms A in A_1, B in B_1 and C in C_1. It transforms O in H. ■

2.1.5 Centers of in- and ex- circles of a triangle

Definition 2.1.26 Let ABC be a triangle. The line α containing the bisector of $\measuredangle BAC$ is called the *interior bisector* through A of the triangle ABC. The line $\alpha^{\perp}$ through A, perpendicular to α, is called the *exterior bisector* through A of the triangle ABC.

Theorem 2.1.27 The three interior bisectors α, β and γ of a triangle ABC are concurrent in a point I. Two exterior bisectors through two vertices are secant in a point belonging to the interior bisector through the third vertex:

α, $\beta^\perp$, $\gamma^\perp$ are concurrent in I_A,

β, $\gamma^\perp$, $\alpha^\perp$ are concurrent in I_B,

γ, $\alpha^\perp$ and $\beta^\perp$ are concurrent in I_C.

The points I, I_A, I_B and I_C are the centers of the circles tangent to the three sides of the triangle. The circle of center I is called the *in-circle* of ABC, the three others are called the *ex-circles* of ABC.

Proof. The points I, I_A, I_B and I_C are equidistant from the sides of the triangle, see Figure 27. ■

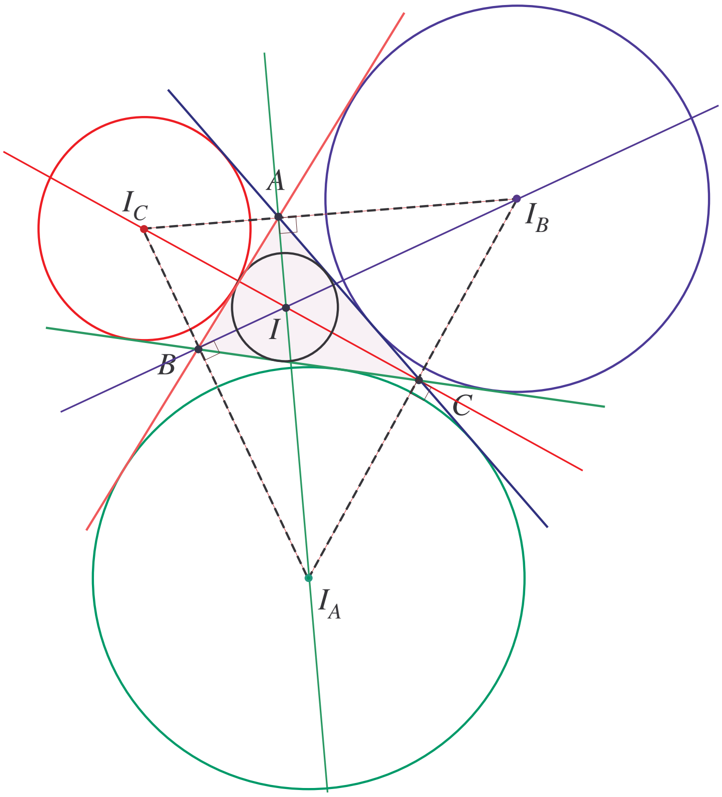

Figure 27: Triangle in- and ex- circles

Proposition 2.1.28 Let us call a = dist(B,C), b = dist(C,A) and c = dist(A,B). The points I, I_A, I_B and I_C have as barycentric coordinates relatively to ABC: (a,b,c), $(-a,b,c)$, $(a,-b,c)$ and $(a,b,-c)$.

Proof. Let us call P the intersection of α with $\overleftrightarrow{BC}$. Remember that the area of a triangle is $\frac{1}{2}$ base $\times$ altitude. The two triangles ABP and APC having the common altitude through A, we have:

$$\frac{\text{dist}(P,B)}{\text{Area}\,(ABP)} = \frac{\text{dist}(P,C)}{\text{Area}\,(APC)}.$$

Since the point P is equidistant from the sides $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, we also have

$$\frac{\text{dist}(A,B)}{\text{Area}\,(ABP)} = \frac{\text{dist}(A,C)}{\text{Area}\,(APC)}.$$

So

$$\frac{\text{dist}(P,B)}{\text{dist}(A,B)} = \frac{\text{dist}(P,C)}{\text{dist}(A,C)} \quad \text{or} \quad \frac{\text{dist}(P,B)}{c} = \frac{\text{dist}(P,C)}{b}$$

which means that $(0,b,c)$ are barycentric coordinates of P. The line $\overleftrightarrow{AP}$ then goes through the point of barycentric coordinates (a,b,c). The three interior bisectors going through this point, it must be the point I. Notice that the intersection P' of $\alpha^{\perp}$ with $\overleftrightarrow{BC}$ is the harmonic conjugate of P relatively to B and C. It follows that $(0,-b,c)$, or as well $(0,b,-c)$, are barycentric coordinates of P'. The point with barycentric coordinates $(a,-b,c)$ then belongs to $\overleftrightarrow{AP'}$. In the same way it belongs to $\overleftrightarrow{CR'}$, the exterior bisector through C. It also belongs to $\overleftrightarrow{BQ}$, the interior bisector through B, since Q admits the barycentric coordinates $(c,0,a)$. Finally the point with barycentric coordinates $(a,-b,c)$ is I_B. ■

2.2 Orthocentric quadrangles associated with a triangle

2.2.1 A, B, C and H

Proposition 2.2.1 If H is the orthocenter of ABC, then A is the orthocenter of HBC, B is the orthocenter of AHC and C is the orthocenter of ABH.

Lemma 2.2.2 Let A, B, C and D be four points. Then:

$$\left.\begin{array}{c} \overleftrightarrow{AC} \perp \overleftrightarrow{BD} \\ \overleftrightarrow{AD} \perp \overleftrightarrow{BC} \end{array}\right\} \quad \Longrightarrow \quad \overleftrightarrow{AB} \perp \overleftrightarrow{CD}.$$

Proof. Let $u := A - B$, $v := A - C$ and $w := A - D$. The trivial formula:

$$u(v-w) + v(w-u) + w(u-v) = 0$$

is equivalent to:

$$(B-A)(D-C) + (C-A)(B-D) + (D-A)(C-B) = 0.$$

If the last two terms of that sum are 0, then the first is also 0. ■

Definition 2.2.3 Four points A, B, C and D form an *orthocentric quadrangle* if

$$\overleftrightarrow{AB} \perp \overleftrightarrow{CD}, \qquad \overleftrightarrow{AC} \perp \overleftrightarrow{BD} \qquad \text{and} \qquad \overleftrightarrow{AD} \perp \overleftrightarrow{BC}.$$

Remark 2.2.4 Following the lemma (2.2.2), if two of the relations are verified, the third is a consequence.

Proposition 2.2.5 The three vertices and the orthocenter of a triangle form an orthocentric quadrangle (see Figure 28).

Definition 2.2.6 Given a triangle ABC the feet of the altitudes are the vertices of a triangle called the *podar triangle* of ABC (see Figure 29).

Proposition 2.2.7 The four triangles one can form out of an orthocentric quadrangle have the same podar triangle (see Figure 30).

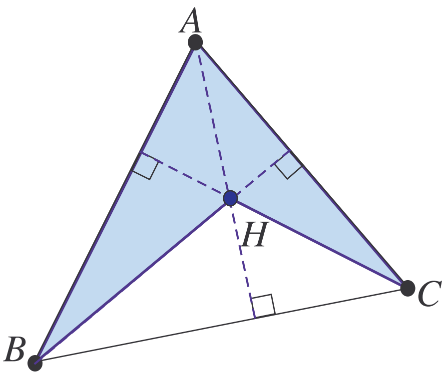

Figure 28: An orthocentric quadrangle associated to a triangle

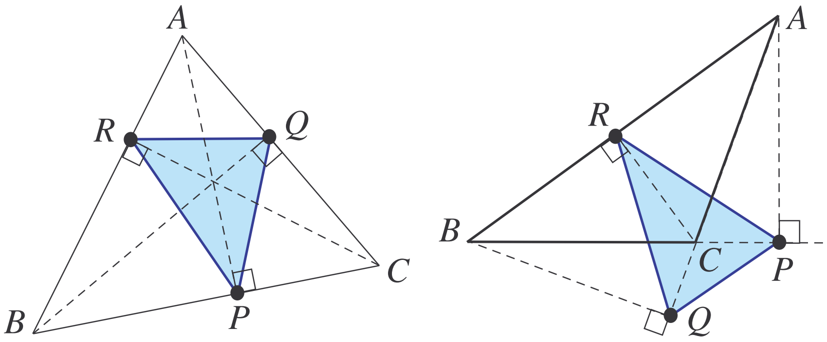

Figure 29: The podar triangle of a triangle

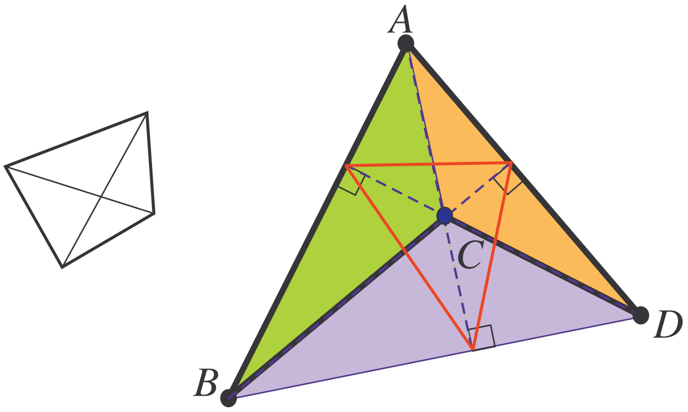

Figure 30: Four triangles ABC, BCD, CDA, DAB of an orthocentric quadrangle

Theorem 2.2.8 Let $ABCD$ be an orthocentric quadrangle and let PQR be its podar triangle. The circle circumscribed PQR goes through the six midpoints of the quadrangle (see Figure 31).

Proof. Take a look at the Figure 32. Let P be the foot of the altitude through A of the triangle ABC, let O be the center of the circumscribed circle to ABC and let G be the centroid of ABC. The central dilatation of center G and with $\kappa = -\frac{1}{2}$ transforms A in A', the midpoint of BC,

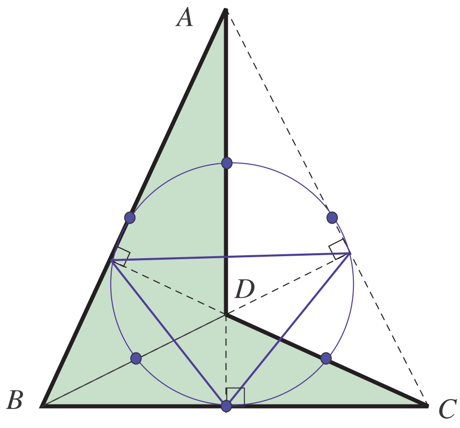

Figure 31: Circle through the six midpoints

the circum-circle to ABC in the circum-circle to $A'B'C'$, D in O and O in O'. It follows that O' is the midpoint of DO. The Thales theorem thus shows that the orthogonal projection of O' on $\overleftrightarrow{BC}$ is the midpoint of PA'. We then have $\text{dist}(O', P) = \text{dist}(O', A')$ and P belongs to the circum-circle to $A'B'C'$. This result proves also that Q and R belongs to that circle. The circles circum PQR and $A'B'C'$ are thus the same. Since ABD has the same podar triangle PQR as ABC, we conclude that the circle circum PQR goes through the midpoints of AB and BD, and so on. ■

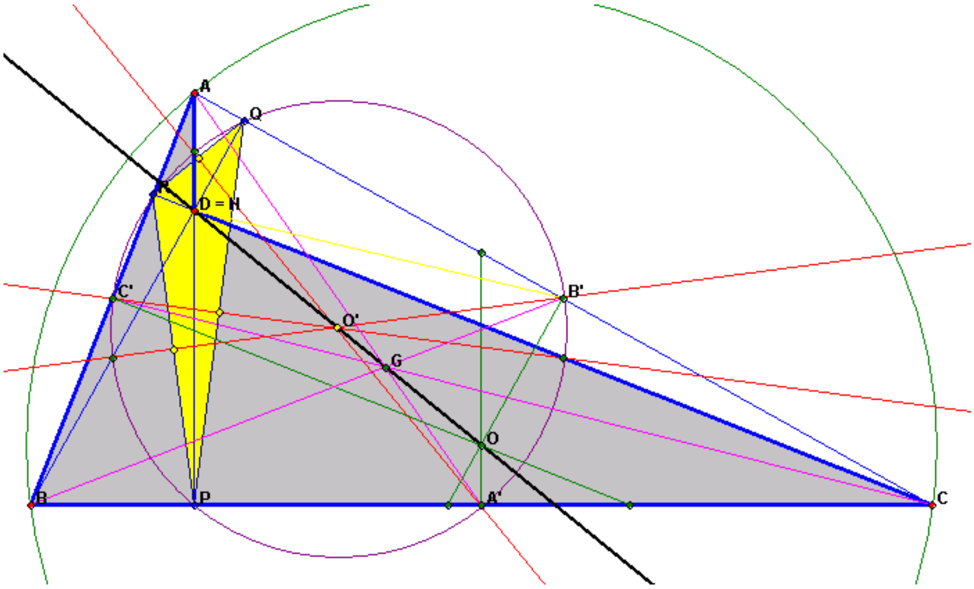

Figure 32: Quadrangle and nine point circle

Quadrangle and nine point circle (link to JavaSketchpad animation)
`http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/9PointCircleQuadrangle.htm`

Definition 2.2.9 Let ABC be a triangle, let A', B' and C' be the midpoints of BC, CA and AB and let PQR be the podar triangle of ABC.The circle through A', B' and C' and through P, Q and R is called the *9-points circle* or the *Euler circle* of ABC.

Triangle special points (unfinished version) (link to JavaSketchpad animation) `http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/TriangleSpecialPoints.htm`

Exercise 2.2.10 Let ABC be a triangle and let H be the orthocenter of ABC. Show that the radii of the circum-circles of HBC, HCA and HAB are equal.

Hint: show first that the radius of the circum-circle to ABC is twice the radius of the Euler circle; show that HBC has the same Euler circle as ABC.

Exercise 2.2.11 Let ABC be a triangle with acute angles. Find the inscribed triangle of minimum perimeter; that means find P belonging to the segment $[BC]$, Q belonging to the segment $[CA]$ and R belonging to the segment $[AB]$ such that $\text{dist}(P, Q) + \text{dist}(Q, R) + \text{dist}(R, P)$ is minimum.

Hint: consider the point P' symmetrical to P relatively to $\overleftrightarrow{AB}$ and P'' symmetrical to P relatively to $\overleftrightarrow{AC}$.

Fix P and minimize $\phi_P(Q, R) := \text{dist}(P', Q) + \text{dist}(Q, R) + \text{dist}(R, P'')$.

Finally minimize relatively to P.

2.2.2 Examples of orthonormal quadrangles: A', B', C', O; I, I_A, I_B, I_C

Exercise 2.2.12 Let ABC be a triangle, let A' be the midpoint of BC, B' be the midpoint of CA, C' be the midpoint of AB, and let O be the center of the circum-scribed circle to ABC. Show that $A'B'C'O$ is an orthocentric quadrangle.

Exercise 2.2.13 Let ABC be a triangle, let I, I_A, I_B and I_C be the centers of the in- and ex-circles of ABC. Show that $II_AI_BI_C$ is an orthocentric quadrangle and determine its podar triangle.

2.2.3 Orthocentric tetrahedron

To generalize results about the triangle to results about the tetrahedron is usually rather easy, but you have to be a little bit careful with the orthocenter, which does not always exist and you have to be very careful with spheres tangent to the four faces of the tetrahedron.

Exercise 2.2.14 Let $ABCD$ be a tetrahedron, we say that H is the *orthocenter* of $ABCD$ if the orthogonal lines to the faces through the opposite vertex are concurrent in H, see Figure 33.

Show that $ABCD$ has an orthocenter if and only if each edge is orthogonal to the opposite one. Such a tetrahedron is called an *orthocentric tetrahedron*.

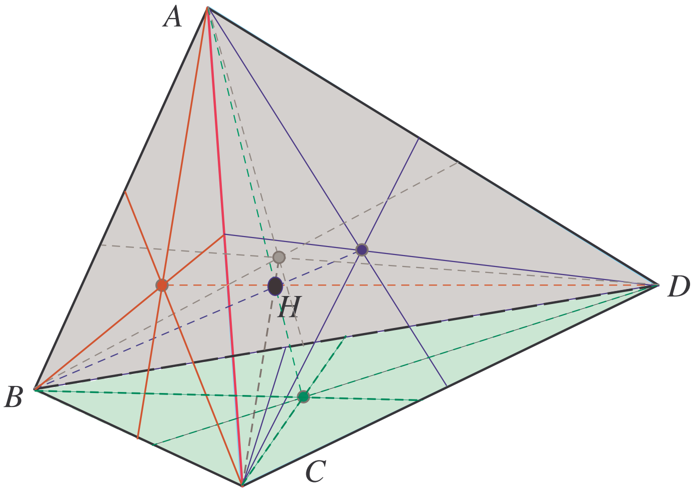

Figure 33: Tetrahedron orthocenter

3 Algebraic description of a euclidean space of dimension 3

3.1 The Clifford algebra $R_{3,0}$

3.1.1 Definition

Definition 3.1.1 The *Clifford algebra* $A := \mathbb{R}_{3,0}$ is the real algebra generated by 1 and by three vectors e_1, e_2 and e_3 such that:

$$\begin{cases} e_1^2 = e_2^2 = e_3^2 = 1 \\ e_1e_2 = -e_2e_1, \qquad e_1e_3 = -e_3e_1, \qquad e_2e_3 = -e_3e_2 \end{cases}$$

Proposition 3.1.2 A is vector space of dimension 8 with basis:

$$1,\ e_1,\ e_2,\ e_3,\ e_{12},\ e_{13},\ e_{23},\ e_{123}$$

where $e_{12} = e_1e_2$, $e_{13} = e_1e_3$, $e_{23} = e_2e_3$ and $e_{123} = e_1e_2e_3$.

Proof. Any product of the form $e_{i_1}e_{i_2}\dots e_{i_n}$ can be reduced to one of the eight elements of A given above. We admit that it is possible to construct an algebra in which these 8 elements are linearly independent. ■

Notation 3.1.3 The element $e_{123} = e_1e_2e_3$ will be denoted by i:

$$i = e_1e_2e_3.$$

The most general element of A may be written:

$$\gamma = \lambda + x + w + \nu i$$

with λ, x_1, x_2, x_3, w_{12}, w_{13}, w_{23} and ν real numbers,

$$\begin{aligned} x &= x_1e_1 + x_2e_2 + x_3e_3, \\ w &= w_{12}e_{12} + w_{13}e_{13} + w_{23}e_{23}. \end{aligned}$$

Definition 3.1.4 λ is called a *scalar*, x a *vector*, w a *bivector* and νi a *pseudo-scalar*.

3.1.2 Computations in $\mathbb{R}_{3,0}$

Proposition 3.1.5 If $x = x_1e_1 + x_2e_2 + x_3e_3$ is a vector then $x^2 = x_1^2 + x_2^2 + x_3^2$ is a positive real number and $x^2 = 0$ if and only if $x = 0$.

Proof. We have

$$\begin{aligned} (x_1e_1 + x_2e_2 + x_3e_3)^2 &= x_1^2e_1^2 + x_2^2e_2^2 + x_3^2e_3^2 + x_1x_2(e_1e_2 + e_2e_1) \\ &\quad + x_1x_3(e_1e_3 + e_3e_1) + x_2x_3(e_2e_3 + e_3e_2), \end{aligned}$$

and since $e_1^2 = 1$, $e_2^2 = 1$, $e_3^2 = 1$ and $e_1e_2 + e_2e_1 = e_1e_3 + e_3e_1 = e_2e_3 + e_3e_2 = 0$, we get

$$(x_1e_1 + x_2e_2 + x_3e_3)^2 = x_1^2 + x_2^2 + x_3^2.$$

■

	1	e_1	e_2	e_3	e_{12}	e_{13}	e_{23}	i
1	1	e_1	e_2	e_3	e_{12}	e_{13}	e_{23}	i
e_1	e_1	1	e_{12}	e_{13}	e_2	e_3	i	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_1$	$-i$	e_3	$-e_{13}$
e_3	e_3	$-e_{13}$	$-e_{23}$	1	i	$-e_1$	$-e_2$	e_{12}
e_{12}	e_{12}	$-e_2$	e_1	i	-1	$-e_{23}$	e_{13}	$-e_3$
e_{13}	e_{13}	$-e_3$	$-i$	e_1	e_{23}	-1	$-e_{12}$	e_2
e_{23}	e_{23}	i	$-e_3$	e_2	$-e_{13}$	e_{12}	-1	$-e_1$
i	i	e_{23}	$-e_{13}$	e_{12}	$-e_3$	e_2	$-e_1$	-1

Table 1: The binary operation table

Proposition 3.1.6 If $w = w_{12}e_{12} + w_{13}e_{13} + w_{23}e_{23}$ is a bivector then $w^2 = -w_{12}^2 - w_{13}^2 - w_{23}^2$ is a negative real number and $w^2 = 0$ if and only if $w = 0$.

Proof. We have

$$\begin{aligned}(w_{12}e_{12} + w_{13}e_{13} + w_{23}e_{23})^2 &= w_{12}^2e_{12}^2 + w_{13}^2e_{13}^2 + w_{23}^2e_{23}^2 + w_{12}w_{13}(e_{12}e_{13} + e_{13}e_{12}) \\ &\quad + w_{12}w_{23}(e_{12}e_{23} + e_{23}e_{12}) + w_{13}w_{23}(e_{13}e_{23} + e_{23}e_{132}),\end{aligned}$$

and since $e_{12}^2 = -1$, $e_{13}^2 = -1$, $e_{23}^2 = -1$ and $e_{12}e_{13} + e_{13}e_{12} = e_{12}e_{23} + e_{23}e_{12} = e_{13}e_{23} + e_{23}e_{13} = 0$, we get

$$(w_{12}e_{12} + w_{13}e_{13} + w_{23}e_{23})^2 = -w_{12}^2 - w_{13}^2 - w_{23}^2.$$

■

Proposition 3.1.7 The center of A is $\mathbb{R} + \mathbb{R}i$.

This means that the elements of A which commute with all elements in A are the scalars and the pseudo-scalars.

Proof. Look at Table 1: the last line and the last column are identical. ■

Huomatkaa! A is not integer: there are elements in A different from 0 but such that their product is 0. For instance if x is a vector such that $x^2 = 1$, then $(1 - x)(1 + x) = 0$, but $1 - x \neq 0$ and $1 + x \neq 0$. You may even find $\gamma \neq 0$ but such that $\gamma^2 = 0$. For example: $\gamma := e_1 + e_{12}$.

Proposition 3.1.8 A vector x is invertible if and only if it is different from 0 and then:

$$x^{-1} = \frac{1}{x^2}\, x.$$

Proposition 3.1.9 A bivector w is invertible if and only if it is different from 0 and then:

$$w^{-1} = \frac{1}{w^2}\, w$$

Proposition 3.1.10 If x, y and z are vectors explicitly given by: $x = x_1e_1 + x_2e_2 + x_3e_3$, $y = y_1e_1 + y_2e_2 + y_3e_3$ and $z = z_1e_1 + z_2e_2 + z_3e_3$, then:

$$\begin{aligned} xy &= x_1y_1 + x_2y_2 + x_3y_3 + (x_1y_2 - x_2y_1)e_{12} + (x_1y_3 - x_3y_1)e_{13} + (x_2y_3 - x_3y_2)e_{23} \\ xyz &= (x_1y_1 + x_2y_2 + x_3y_3)\,z - (x_1z_1 + x_2z_2 + x_3z_3)\,y + (y_1z_1 + y_2z_2 + y_3z_3)\,x \\ &\quad + (x_1y_2z_3 - x_1y_3z_2 - x_2y_1z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1)\,i \end{aligned}$$

Proof. Just do the computations … with care. ■

3.1.3 Reversion or principal involution

We could write the products from right to left instead of as usual from left to right. The theory would be the same, but all the formulas would look different. We have thus a bijection of A into itself.

Definition 3.1.11 The linear bijection from A to A, denoted by$\widetilde{\ }$, such that:

$$\begin{cases} \forall\, i \in \{1,2,3\} : & \widetilde{e_i} = e_i \\ \text{and} & \\ \forall\, \gamma_1, \gamma_2 \in A : & \widetilde{\gamma_1\gamma_2} = \widetilde{\gamma_2}\widetilde{\gamma_1} \end{cases}$$

is called the *reversion* or *principal involution* of A.

Proposition 3.1.12 $\widetilde{(\widetilde{\gamma})} = \gamma$. If λ is a scalar $\widetilde{\lambda} = \lambda$, if x is a vector $\widetilde{x} = x$, if w is a bivector $\widetilde{w} = -w$, if νi is a pseudo-scalar $\widetilde{(\nu i)} = -\nu i$.

Proof. $\widetilde{e_1e_2} = e_2e_1 = -e_1e_2$, and in the same way $\widetilde{e_1e_3} = -e_1e_3$ and $\widetilde{e_2e_3} = -e_2e_3$, so $\widetilde{w} = -w$. For the pseudo-scalars we have:

$$\widetilde{i} = \widetilde{e_1e_2e_3} = e_3e_2e_1 = -e_2e_3e_1 = e_2e_1e_3 = -e_1e_2e_3 = -i.$$

■

3.1.4 Quaternions

Definition 3.1.13 We write $I := e_{23}$, $J := e_{31}$ and $K := e_{12}$. The elements of the form $\lambda + \xi I + \eta J + \varsigma K$ are called *quaternions*. The set of quaternions is denoted by $\mathbb{H}$.

Proposition 3.1.14 $\mathbb{H}$ is a subalgebra of A.

Proof. $IJ = -JI = K$, $JK = -KJ = I$ and $KI = -IK = J$. So the product of two quaternions is still a quaternion. ■

Proposition 3.1.15 If q is a quaternion, then $q\widetilde{q} = \widetilde{q}q$ is a real positive number, strictly positive if and only if $q \neq 0$. Every quaternion distinct from 0 is invertible and

$$q^{-1} = \frac{1}{q\widetilde{q}}\, q.$$

Proof. If $q = \lambda + \xi I + \eta J + \varsigma K$, we have $\widetilde{q} = \lambda - \xi I - \eta J - \varsigma K$ and $q\widetilde{q} = \lambda^2 + \xi^2 + \eta^2 + \varsigma^2$. ■

3.2 Geometrical interpretation

3.2.1 Scalars, vectors, bivectors and pseudo-scalars

Scalars: = numbers

Vectors: = points or vectors in a three-dimensional euclidean space E in which an orthonormal frame (O, e_1, e_2, e_3) has been chosen. We ought to show that everything we are going to do is independent of choice of this frame. We will admit that result.

Bivectors: If u and v are vectors that are not collinear, uv is a scalar plus a bivector. The scalar is the ordinary inner product $u \cdot v$ and the bivector keeps the information of the plane defined by u and v and the oriented area of the parallelogram build on u and v.

Pseudo-scalars: The set of pseudo-scalars is a one-dimensional object. The pseudo-scalars do not change in absolute value when you change the frame of reference, but their signs change into opposite signs if we change the orientation of the frame. Practically a pseudo-scalar can be considered as an oriented volume since for three vectors x, y and z we have:

$$xyz = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} i = \det(x, y, z)\, i$$

3.2.2 Orthogonality and parallelism

Proposition 3.2.1 Two vectors x and y are parallel if and only if they commute, that is if $xy = yx$; and they are orthogonal if and only if they anticommute, that is if $xy = -yx$.

Proof. Let $x = x_1e_1 + x_2e_2 + x_3e_3$ and $y = y_1e_1 + y_2e_2 + y_3e_3$. They are parallel if and only if $x_1y_2 - x_2y_1 = x_1y_3 - x_3y_1 = x_2y_3 - x_3y_2 = 0$ and they are orthogonal if and only if $x_1y_1 + x_2y_2 + x_3y_3 = 0$. Let us compute:

$$xy - yx = (x_1y_2 - x_2y_1)e_{12} + (x_1y_3 - x_3y_1)e_{13} + (x_2y_3 - x_3y_2)e_{23},$$

so we have the first part. Let us compute $xy + yx = x_1y_1 + x_2y_2 + x_3y_3$. We get the last part. ■

Proposition 3.2.2 A vector x and a bivector w are parallel if and only if they anticommute, they are orthogonal if and only if they commute.

Proof. Let us write $w := yz$, where y and z are orthogonal vectors. Choose t such that (y, z, t) is a direct orthogonal basis for E. We can write $x = \alpha y + \beta z + \gamma t$, with α, β and γ real numbers. We have

$$wx = yz(\alpha y + \beta z + \gamma t) = y(-\alpha y + \beta z - \gamma t)z = (-\alpha y - \beta z + \gamma t)yz.$$

So $wx + xw = 0$ if and only if $\gamma = 0$ and $wx - xw = 0$ if and only if $\alpha = \beta = 0$. ■

Proposition 3.2.3 If u is a vector, the plane orthogonal to u is characterized by the bivector iu: a vector x belongs to the plane orthogonal to u if and only if it is parallel to iu and a bivector w' is orthogonal to u if and only if $w' \in \mathbb{R}\, iu$.

Proof. A vector x is orthogonal to u if and only if $xu = -ux$, or $xiu = -iux$, which means that x is parallel to the bivector iu. ■

3.2.3 Angles between vectors

Let x and y be two unit vectors in some plane. Let us take an orthonormal basis (u, v) in that plane. That means that $u^2 = 1$, $v^2 = 1$ and $uv = -vu$. We have:

$$\begin{cases} x = (\cos\alpha)u + (\sin\alpha)v \\ y = (\cos\beta)u + (\sin\beta)v \end{cases}$$

and $\theta = \beta - \alpha$ is the angle between $x$and y. By direct computation, we get:

$$xy = \cos\alpha\cos\beta + \sin\alpha\sin\beta + uv(\cos\alpha\sin\beta - \sin\alpha\cos\beta) = \cos\theta + w\sin\theta,$$

where $w = uv$ is the bivector of norm -1 describing the plane in which are x and y, with a choice of orientation. This choice is the one in which the sens is positive from u to v or in other words, the orientation for which the basis (u, v) is direct.

Proposition 3.2.4 If x and y are two unit vectors, we have

$$xy = \exp(w\theta),$$

where θ is the measure of the oriented angle of these two vectors in the plane defined and oriented by the bivector w.

Proof. By definition of exp we have:

$$\exp(w\theta) = 1 + w\theta + \frac{1}{2!}(w\theta)^2 + \ldots + \frac{1}{n!}(w\theta)^n + \ldots$$

and since $w^2 = -1$, we have

$$\exp(w\theta) = 1 + w\theta - \frac{1}{2!}(\theta)^2 + \ldots + (-1)^n\frac{1}{2n!}(\theta)^{2n} + (-1)^n\frac{1}{(2n+1)!}(\theta)^{2n+1}w + \ldots$$

or

$$\exp(w\theta) = \cos\theta + w\sin\theta = xy.$$

■

3.2.4 Vector calculus

Definition 3.2.5 The *inner product* of two vectors x and y is:

$$x \cdot y = \frac{1}{2}(xy + yx)$$

and the *vectorial product* or *cross product* is:

$$x \wedge y = x \times y = -i\frac{1}{2}(xy - yx)$$

Remark 3.2.6 With these definition, we are coming back to the usual definitions in vector calculus: if x and y are vectors explicitly given by: $x = x_1e_1 + x_2e_2 + x_3e_3$ and $y = y_1e_1 + y_2e_2 + y_3e_3$, then

$$\begin{aligned} x \cdot y &= x_1y_1 + x_2y_2 + x_3y_3 \\ x \wedge y = x \times y &= (x_1y_2 - x_2y_1)e_3 - (x_1y_3 - x_3y_1)e_2 + (x_2y_3 - x_3y_2)e_1 \\ x \wedge y = x \times y &= (x_2y_3 - x_3y_2)e_1 + (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3 \end{aligned}$$

Proposition 3.2.7 $x \cdot (y \wedge z) = \det(x, y, z)$ and $x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z$.

Proof. We can rewrite the former result

$$\begin{aligned} xyz &= (x_1y_1+x_2y_2+x_3y_3)\,z-(x_1z_1+x_2z_2+x_3z_3)\,y+(y_1z_1+y_2z_2+y_3z_3)\,x \\ &\quad +(x_1y_2z_3-x_1y_3z_2-x_2y_1z_3+x_2y_3z_1+x_3y_1z_2-x_3y_2z_1)\,i \end{aligned}$$

as $xyz=(x\cdot y)\,z-(x\cdot z)\,y+(y\cdot z)\,x+\det(x,y,z)\,i$. Thus we get:

$$\begin{aligned} x\cdot(y\wedge z) &= \frac{1}{2}(x\frac{-i}{2}(yz-zy)+\frac{-i}{2}(yz-zy)x)=\frac{-i}{4}(xyz-xzy+yzx-zyx) \\ &= \frac{-i}{4}(i\det(x,y,z)+(x\cdot y)z-(x\cdot z)y+(y\cdot z)x \\ &\quad -(i\det(x,z,y)+(x\cdot z)y-(x\cdot y)z+(z\cdot y)x) \\ &\quad +(i\det(y,z,x)+(y\cdot z)x-(y\cdot x)z+(z\cdot x)y) \\ &\quad -(i\det(y,x,z)+(y\cdot x)z-(y\cdot z)x+(x\cdot z)y) \\ &= \det(x,y,z). \end{aligned}$$

In the same way, you obtain:

$$x\wedge(y\wedge z)=\frac{(-i)^2}{4}(xyz-xzy-yzx+zyx)=(x\cdot z)y-(x\cdot y)z.$$

■

3.3 Transformations in the euclidean space E

3.3.1 Projections and symmetries relatively to a vector

Proposition 3.3.1 Let u be a vector. For any vector x, let us denote by $x_\parallel$ the orthogonal projection on u and by $x_\perp$ the orthogonal projection on the plane orthogonal to u, see Figure 34. Then:

$$\begin{aligned} x_\parallel &= \frac{1}{2}(x+u^{-1}xu) \\ x_\perp &= \frac{1}{2}(x-u^{-1}xu) \end{aligned}$$

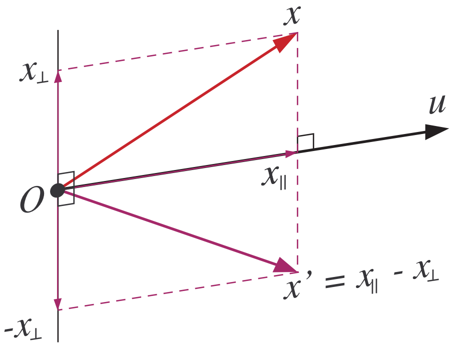

Figure 34: Projections relative a vector

Proof. First, $x = x_{\|} + x_{\perp}$. To translate that $x_{\|}$ is parallel to u, we write that they commute: $x_{\|}u = ux_{\|}$ and to translate that $x_{\perp}$ is orthogonal to u, that they anticommute: $x_{\perp}u = -ux_{\perp}$. So: $ux + xu = 2ux_{\|}$ and $ux - xu = 2ux_{\perp}$. Multiplying on the right by u^{-1} we get the result. ■

Remark 3.3.2 Do not forget that $u^{-1} = \dfrac{1}{u^2}u$.

Proposition 3.3.3 The vector x' symmetrical to x relatively to the vector u is:

$$x' = u^{-1}xu = uxu^{-1}$$

and if u is of norm 1, then $x' = uxu$.

Proof. By definition of symmetry $x' = x_{\|} - x_{\perp}$. ■

Corollary 3.3.4 Let u be a **unitary** vector directing a line d which goes through O. The vector x' symetrical to x relatively to the line d is:

$$x' = uxu.$$

3.3.2 Projections and symmetries relatively to a bivector

Proposition 3.3.5 Let w be a bivector. For any vector x, let us denote by $x_{\|}$ the orthogonal projection on w and by $x_{\perp}$ the orthogonal projection on the line orthogonal to w, see Figure 35. Then:

$$\begin{aligned} x_{\|} &= \frac{1}{2}(x - w^{-1}xw) \\ x_{\perp} &= \frac{1}{2}(x + w^{-1}xw) \end{aligned}$$

Proof. Again $x = x_{\|} + x_{\perp}$. To translate that $x_{\|}$ is parallel to w, we write that they anticommute: $x_{\|}w = -wx_{\|}$ and to translate that $x_{\perp}$ is orthogonal to w, that they commute: $x_{\perp}w = wx_{\perp}$. So: $wx + xw = 2wx_{\perp}$ and $wx - xw = 2wx_{\|}$. Multiplying on the right by w^{-1} we get the result. ■

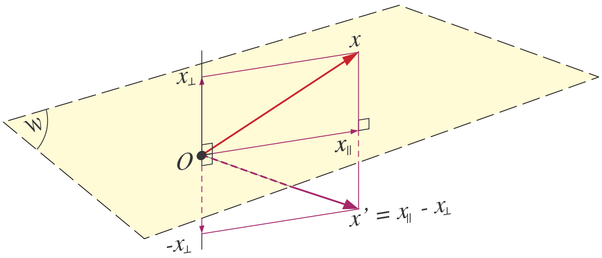

Figure 35: Projections relative a bivector

Remark 3.3.6 If w is a bivector different from 0 it is invertible, we have then

$$w^2 = -w_{12}^2 - w_{13}^2 - w_{23}^2 < 0 \text{ and } w^{-1} = \frac{1}{w^2}w.$$

Proposition 3.3.7 The vector x' symmetrical to x relatively to the bivector w is:

$$x' = -w^{-1}xw = -wxw^{-1}$$

and if w is unitary in the sens that $w^2 = -1$, then:

$$x' = wxw.$$

Proof. By definition of symmetry $x' = x_{\parallel} - x_{\perp} = -w^{-1}xw$. But $w^{-1} = \frac{1}{w^2}\,w$ and $\frac{1}{w^2}$ is a scalar commuting with all the elements, so:

$$x' = -w^{-1}xw = -\frac{1}{w^2}\,wxw = -wx\frac{1}{w^2}\,w = -wxw^{-1}.$$

If $w^2 = -1$, then $x' = wxw$. ■

3.3.3 Translation

Translations are mappings of the form $x \mapsto x + a$, where a is a vector.

3.3.4 Rotations around an axis going through O

A rotation can be considered as the composition of two reflections along two planes, the axis of the rotation being the intersection of these two planes and the angle being the double of the angle between these planes. Let u be a unit vector orthogonal to the first plane. The plane itself is represented by the bivector iu. Let v be a unit vector orthogonal to the second plane. The plane itself is represented by the bivector iv.

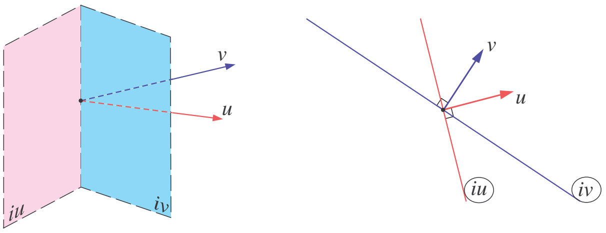

Figure 36: Rotation around an axis through O

By the first reflection a vector x will be transformed in $x_1 = (iu)x(iu)$, by the second reflection x_1 will be transformed in $x' = (iv)x_1(iv)$. The rotation is thus:

$$x \mapsto x' = (iv)(iu)x(iu)(iv) = vuxuv$$

If u and v are collinear, the transformation is the identity. If they are not collinear the vector

$$d = \frac{1}{\sqrt{(u \wedge v)^2}}\,u \wedge v$$

is defined and $uv = \exp(id\frac{\theta}{2})$, where $\frac{\theta}{2}$ is the measure of the oriented angle between the lines $\mathbb{R}u$ and $\mathbb{R}v$ and θ is the measure of the angle (angle of rays) of the rotation, the orientation of the plane $\langle u, v\rangle$ being defined by the vector d orthogonal to it.

Proposition 3.3.8 Let d be a vector and L the line through O directed by the vector d and let θ be a measure of an oriented angle of rays. The rotation R of axis L and (measure of) angle of rotation θ in a plane orthogonal to d and oriented by d is the transformation:

$$x \mapsto x' = \exp\left(-id\frac{\theta}{2}\right)\, x \,\exp\left(id\frac{\theta}{2}\right).$$

Proof. It remains simply to note that $vu = \widetilde{uv} = \widetilde{\exp(id\frac{\theta}{2})} = \exp(-id\frac{\theta}{2})$. ■

Corollary 3.3.9 The composition of a rotation R_1 followed by a rotation R_2 denoted is a rotation $R = R_2 \circ R_1$. If we denote unitary vectors and angles of these rotations by d_1, θ_1, d_2, θ_2,d and θ, then it easy to compute d and θ, knowing d_1, θ_1, d_2 and θ_2:

$$\exp\left(id\frac{\theta}{2}\right) = \exp\left(id_1\frac{\theta_1}{2}\right)\exp\left(id_2\frac{\theta_2}{2}\right)$$

or

$$\cos\left(\frac{\theta}{2}\right) + id\sin\left(\frac{\theta}{2}\right) = \left(\cos\left(\frac{\theta_1}{2}\right) + id_1\sin\left(\frac{\theta_1}{2}\right)\right)\left(\cos\left(\frac{\theta_2}{2}\right) + id_2\sin\left(\frac{\theta_2}{2}\right)\right).$$

Let us end this computation. Remember that $d_1d_2 = d_1 \cdot d_2 + id_1 \wedge d_2$, where $d_1 \cdot d_2$ is a scalar and $id_1 \wedge d_2$ is a vector. The scalar parts of the above equality have to be equal:

$$\cos\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) - d_1 \cdot d_2 \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)$$

and the bivectorial parts have also to be equal:

$$id\sin\left(\frac{\theta}{2}\right) = id_1\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + id_2\sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) - id_1 \wedge d_2 \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right).$$

3.3.5 Rotations around an axis going through a

$$x \longmapsto x' = \exp\left(-id\frac{\theta}{2}\right)(x-a)\exp\left(id\frac{\theta}{2}\right) + a$$

Remark 3.3.10 For d and θ given, we guess that the rotations defined with a and with a' will be the same if the line through a and a' is parallel to d or if $a' - a$ is parallel to d, that is if $a' - a$ and d commute. It is easy to see that it is the case, because then $a' - a$ commutes with $\exp(id\frac{\theta}{2})$ and:

$$\begin{aligned}
&\exp\left(-id\frac{\theta}{2}\right)(x-a')\exp\left(id\frac{\theta}{2}\right) + a' = \exp\left(-id\frac{\theta}{2}\right)(x - a - (a'-a))\exp\left(id\frac{\theta}{2}\right) + a' \\
&= \exp\left(-id\frac{\theta}{2}\right)(x-a')\exp\left(id\frac{\theta}{2}\right) - (a'-a)\exp\left(-id\frac{\theta}{2}\right)(\exp\left(id\frac{\theta}{2}\right) + a' \\
&= \exp\left(-id\frac{\theta}{2}\right)(x-a')\exp\left(id\frac{\theta}{2}\right) + a.
\end{aligned}$$

3.3.6 Inversions

Inversion with center O and power k^2:

$$x \longmapsto x' = k^2 x^{-1}.$$

Inversion with center c and power k^2:

$$x \longmapsto x' = k^2 (x - c)^{-1} + c,$$

or:

$$(x' - c)(x - c) = k^2.$$

4 An example of a real affine plane

4.1 The affine plane $\mathcal{P}$

Definition 4.1.1 Let $\mathcal{P}$ be the set of sequences of real numbers $M = (M_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}: \qquad M_{n+2} - M_{n+1} - M_n = n.$$

We call points the elements of $\mathcal{P}$.

Remark 4.1.2 The set $\mathcal{P}$ is a subset of the linear space of all sequences of real numbers $\mathbb{R}^{\mathbb{N}}$, but it is not a subspace since if we multiply by 2 a point of $\mathcal{P}$ we do not get a point belonging to $\mathcal{P}$.

Example 4.1.3 Examples of points

$$\begin{aligned} A &= (1,0,1,2,5,10,19,34,\ldots) \\ B &= (0,1,1,3,6,12,22,39,\ldots) \\ C &= (1,1,2,4,8,15,27,47,\ldots) \end{aligned}$$

The middles of the segments $[B,C]$, $[C,A]$ and $[A,B]$ are

$$\begin{aligned} A' &= \left(\tfrac{1}{2},1,\tfrac{3}{2},\tfrac{7}{2},7,\tfrac{27}{2},\tfrac{49}{2},43,\ldots\right) \\ B' &= \left(1,\tfrac{1}{2},\tfrac{3}{2},3,\tfrac{13}{2},\tfrac{25}{2},23,\tfrac{81}{2},\ldots\right) \\ C' &= \left(\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{5}{2},\tfrac{11}{2},11,\tfrac{41}{2},\tfrac{73}{2},\ldots\right) \end{aligned}$$

The centroid of the triangle ABC is

$$G = \left(\tfrac{2}{3},\tfrac{2}{3},\tfrac{4}{3},3,\tfrac{19}{3},\tfrac{37}{3},\tfrac{68}{3},40,\ldots\right).$$

Definition 4.1.4 Let $M = (M_n)_{n\in\mathbb{N}}$ and $N = (N_n)_{n\in\mathbb{N}}$ be two points in $\mathcal{P}$, we define $\overrightarrow{u} = \overrightarrow{MN}$ by $\overrightarrow{u} := (u_n)_{n\in\mathbb{N}}$ and

$$\forall n \in \mathbb{N}: \qquad u_n = N_n - M_n.$$

4.2 The vectorial plane

Proposition 4.2.1 If M and N belong to $\mathcal{P}$, then $(u_n)_{n\in\mathbb{N}} = \overrightarrow{MN}$ satisfies

$$\forall n \in \mathbb{N}: \qquad u_{n+2} - u_{n+1} - u_n = 0$$

and $\overrightarrow{\mathcal{P}}$ is a linear space.

Proof. Substracting the equalities $N_{n+2} - N_{n+1} - N_n = n$ and $M_{n+2} - M_{n+1} - M_n = n$, we get using $N_k - M_k = u_k$:

$$\forall n \in \mathbb{N}: \qquad u_{n+2} - u_{n+1} - u_n = 0.$$

To show that $\overrightarrow{\mathcal{P}}$ is a linear space, we show that it is a subspace of $\mathbb{R}^{\mathbb{N}}$. This follows from the fact that the second members of the equalities above are all 0. Thus if $\overrightarrow{u} = (u_n)_{n\in\mathbb{N}}$ is in $\overrightarrow{\mathcal{P}}$, then for any real number λ, $\lambda\overrightarrow{u}$ is in $\overrightarrow{\mathcal{P}}$ and if $\overrightarrow{u}$ and $\overrightarrow{u'}$ are in $\overrightarrow{\mathcal{P}}$, then $\overrightarrow{u} + \overrightarrow{u'}$ is in $\overrightarrow{\mathcal{P}}$. ■

Definition 4.2.2 The linear space $\overrightarrow{\mathcal{P}}$ is called a *vectorial plane* and $\mathcal{P}$ is called an *affine plane*. We says that $\overrightarrow{\mathcal{P}}$ is the vectorial plane *associated to* the affine plane $\mathcal{P}$.

We have to justify the use of the word "plane" which suggests that the dimension of the vectorial plane $\overrightarrow{\mathcal{P}}$ is 2.

Definition 4.2.3 Let $\overrightarrow{i} = (i_n)_{n\in\mathbb{N}}$ be the element of $\overrightarrow{\mathcal{P}}$ defined by

$$i_0 = 1 \quad \text{and} \quad i_1 = 0.$$

Definition 4.2.4 Let $\overrightarrow{j} = (j_n)_{n\in\mathbb{N}}$ be the element of $\overrightarrow{\mathcal{P}}$ defined by

$$j_0 = 0 \quad \text{and} \quad j_1 = 1.$$

Remark 4.2.5 Thus we have:

$$\begin{aligned} \overrightarrow{i} &= (1,0,1,1,2,3,\ 5,\ 8,13,21,34,\ 55,\ldots) \\ \overrightarrow{j} &= (0,1,1,2,3,5,\ 8,13,21,34,55,\ 89,\ldots) \\ \overrightarrow{i}+\overrightarrow{j} &= (1,1,2,3,5,8,13,21,34,55,89,144,\ldots) \end{aligned}$$

One can notice that $\overrightarrow{i}+\overrightarrow{j}$ is the Fibonacci sequence and $\overrightarrow{i}$ and $\overrightarrow{j}$ are also the same sequence after 2 terms or 1 term.

Proposition 4.2.6 The couple of vectors $(\overrightarrow{i},\overrightarrow{j})$ is a basis of $\overrightarrow{\mathcal{P}}$.

Proof. Let $\overrightarrow{u}$ be a vector belonging to $\overrightarrow{\mathcal{P}}$. We show by strong induction on n that

$$\forall n \in \mathbb{N}: \quad u_n = u_0 i_n + u_1 j_n$$

It is true for $n = 0$ and $n = 1$. For $n \geq 2$, let us suppose that it is true for all $k < n$. Then

$$u_n = u_{n-1} + u_{n-2} = u_0 i_{n-1} + u_1 j_{n-1} + u_0 i_{n-2} + u_1 j_{n-2} = u_0 i_n + u_1 j_n.$$

From that relation we deduce $\overrightarrow{u} = u_0\overrightarrow{i} + u_1\overrightarrow{j}$ which proves that the couple of vectors $(\overrightarrow{i},\overrightarrow{j})$ generates $\overrightarrow{\mathcal{P}}$. To show that they are independent, suppose $\alpha\overrightarrow{i}+\beta\overrightarrow{j} = \overrightarrow{0}$. Then we have $(\beta,\alpha,\beta{+}\alpha,\ldots) = (0,0,0,\ldots)$ and thus $\alpha = \beta = 0$. ■

Corollary 4.2.7 The dimension of the linear space $\overrightarrow{\mathcal{P}}$ is 2.

4.3 Explicit elements in the vectorial plane

Definition 4.3.1 The *golden number* is the positive number φ such that

$$\varphi^{-1} = \varphi - 1.$$

An explicit computation gives

$$\varphi = \frac{1+\sqrt{5}}{2} \simeq 1{,}61803398875\ldots$$

The other solution of the equation $x^2 - x - 1 = 0$ is

$$\psi = \frac{1-\sqrt{5}}{2} \simeq -0{,}61803398875\ldots$$

Notice that $\varphi + \psi = 1$ and $\varphi\psi = -1$.

Proposition 4.3.2 The sequences $(1, \varphi, \varphi^2, \varphi^3, \dots, \varphi^n, \dots)$ and $(1, \psi, \psi^2, \psi^3, \dots, \psi^n, \dots)$ belong to $\overrightarrow{\mathcal{P}}$.

Proof. Since $\varphi^2 = \varphi + 1$, we have for all n the identity $\varphi^{n+2} = \varphi^{n+1} + \varphi^n$. The same argument holds for ψ. ■

Notation 4.3.3 Let us put

$$\begin{aligned} \overrightarrow{i'} &= (\varphi^n)_{n\in\mathbb{N}} \\ \overrightarrow{j'} &= (\psi^n)_{n\in\mathbb{N}} \end{aligned}$$

Proposition 4.3.4 The couple $(\overrightarrow{i'}, \overrightarrow{j'})$ is a basis of $\overrightarrow{\mathcal{P}}$ and

$$\left\{ \begin{array}{rcl} \overrightarrow{i'} &=& \overrightarrow{i} + \frac{1+\sqrt{5}}{2}\overrightarrow{j} \\ \overrightarrow{j'} &=& \overrightarrow{i} + \frac{1-\sqrt{5}}{2}\overrightarrow{j} \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{rcl} \overrightarrow{i} &=& \frac{5-\sqrt{5}}{10}\overrightarrow{i'} + \frac{5+\sqrt{5}}{10}\overrightarrow{j'} \\ \overrightarrow{j} &=& \frac{\sqrt{5}}{5}\overrightarrow{i'} - \frac{\sqrt{5}}{5}\overrightarrow{j'} \end{array} \right.$$

Proof. The first equalities follow from $i'_0 = 1$, $i'_1 = \varphi = \frac{1+\sqrt{5}}{2}$, $j'_0 = 1$ and $j'_1 = \psi = \frac{1-\sqrt{5}}{2}$. To get the second equalities, inverse the matrix or solve the system. ■

As an immediate consequence we get following theorem.

Theorem 4.3.5 The sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\left\{ \begin{array}{ll} \forall n \in \mathbb{N}: & u_{n+2} - u_{n+1} - u_n = 0 \\ u_0 = a & \\ u_1 = b & \end{array} \right.$$

is given by

$$\forall n \in \mathbb{N}: \qquad u_n = \left(\frac{5-\sqrt{5}}{10}a + \frac{\sqrt{5}}{5}b\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5+\sqrt{5}}{10}a - \frac{\sqrt{5}}{5}b\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

4.4 Back to the affine plane

We need to choose one point Ω in $\mathcal{P}$. For instance

$$\Omega_n = -n - 1$$

This is a point of $\mathcal{P}$ since $-(n+2) - 1 - \big(-(n+1) - 1\big) - (-n-1) = n$. As a consequence of preceding theorem we get the following one.

Theorem 4.4.1 The sequence $(M_n)_{n\in\mathbb{N}}$ such that

$$\left\{ \begin{array}{ll} \forall n \in \mathbb{N} & M_{n+2} - M_{n+1} - M_n = n \\ M_0 = \alpha & \\ M_1 = \beta & \end{array} \right.$$

is given for all by $n \in \mathbb{N}$

$$M_n = -n-1+\left(\frac{5-\sqrt{5}}{10}(\alpha+1) + \frac{\sqrt{5}}{5}(2+\beta)\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5+\sqrt{5}}{10}(1+\alpha) - \frac{\sqrt{5}}{5}(2+\beta)\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Question. How does M_n behave when $n \to +\infty$ if $\alpha = 3$ and $\beta = -2\sqrt{5}$?

5 How to define oriented and unoriented angles in a euclidean plane

5.1 How to use oriented angles of lines

5.1.1 The rules

In the whole chapter we suppose given a euclidean plane $\mathcal{P}$ and we denote by $\mathcal{D}$ the set of lines of $\mathcal{P}$. Given a couple of lines $(a, b) \in \mathcal{D} \times \mathcal{D}$ we will denote by $\langle a, b\rangle$ the oriented angle of these line, starting at a and ending at b. We want these angles to verify for any lines a, b and c:

$$\begin{array}{rcl} \langle a,b\rangle & = & \langle a,c\rangle + \langle c,b\rangle \\ \langle a,b\rangle & = & -\langle b,a\rangle \\ \langle a,b\rangle & = & 0 \quad \text{if and only if} \quad a \parallel b \end{array}$$

and more generally, we want that

> $\langle a, b\rangle = \langle f(a), f(b)\rangle$ for any direct similarity f. The direct similarities are the rotations, the translations, the central dilations and the compositions of these;
>
> $\langle a, b\rangle = -\langle g(a), g(b)\rangle$ for any indirect similarity g. The indirect similarities are the compositions of an odd number of reflections with one central dilation.

We denote the set of oriented angles of lines by OAL . We will see that OAL with the operation $+$ is a group (see Figure 37). Let us just admit it for the moment.

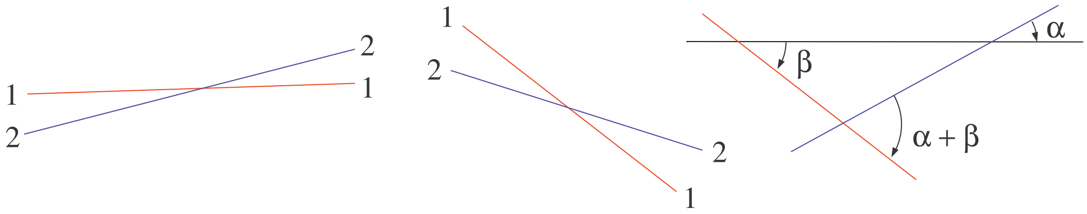

Figure 37: Lines and angles

5.1.2 Solving the equation $2x = \alpha$

Let us consider two perpendicular lines a and b as in Figure 38.

If we do the reflection in b, following our rules we get $\langle a, b\rangle = -\langle a, b\rangle$, since a and b are invariant in this indirect similarity. As a consequence we have

$$\langle a, b\rangle + \langle a, b\rangle = 0.$$

We know that there is always a similarity transforming two perpendicular lines in two others. So there is one and only one straight angle in OAL . Let us call it δ. We have thus found a solution different from 0 to the equation in x in OAL : $2x = 0$. Of course 0 is also a solution. The fact that 0 is a solution is equivalent to the statement $\langle a, a\rangle + \langle a, a\rangle = \langle a, a\rangle$.

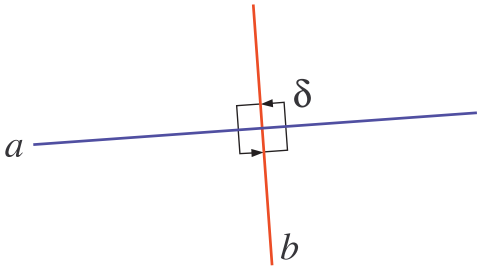

Figure 38: Two perpendicular lines

Exercise 5.1.1 Is it true or false that $\delta = -\delta$. Give a pictorial explanation.

We have found two distinct solutions to the equation $2x = 0$, which are 0 and δ. Let us admit that there are no other and let us look for the equation $2x = \alpha$, where α is a given element of OAL. If x is a solution, then $x + \delta$ is also a solution. Conversely if we have two solutions x_1 and x_2, then $2(x_1 - x_2) = 0$, and so $x_1 - x_2 \in \{0, \delta\}$. If there is one solution there is always another one which differs from the first one by a right angle.

Now, given two lines a and c (see Figure 39), intersecting in a point O, there are two lines through O – let us call them b_1 and b_2 – such that a reflection in any of them exchanges a and c.

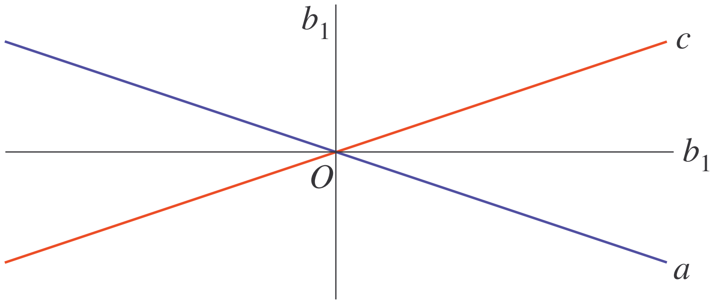

Figure 39: Two lines reflected

Let us choose a and c such that $\langle a, c\rangle = \alpha$. Since reflections are indirect similarities we get then, for $i = 1$ or 2: $\langle a, b_i\rangle = -\langle c, b_i\rangle$ and since $\langle a, b_i\rangle + \langle b_i, c\rangle = \langle a, c\rangle$, we get $2\langle a, b_i\rangle = \langle a, c\rangle$. The lines b_i are called the *bisectors* of the "angle" formed by a and c.

Exercise 5.1.2 Let τ and τ' be the solutions distinct from 0 of the equation $3x = 0$ in OAL. Show that $\tau' = 2\tau$ and $\tau = 2\tau'$. Show that $25(\tau - \delta) = \tau + \delta$.

5.1.3 The fundamental theorem for cocyclicity

Definition 5.1.3 For points A, B, C and D are said to be *cocyclical* if they are collinear or if they belong to a common circle.

Theorem 5.1.4 Four points A, B, C and D are cocyclical if and only if $\langle \overleftrightarrow{CA}, \overleftrightarrow{CB}\rangle = \langle \overleftrightarrow{DA}, \overleftrightarrow{DB}\rangle$ (see Figure 41).

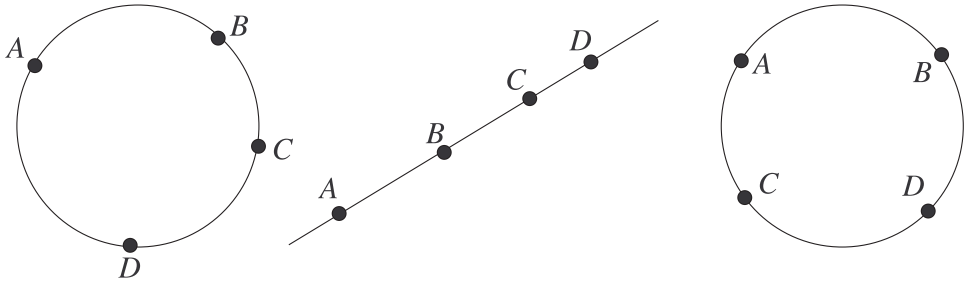

Figure 40: Cocyclic points

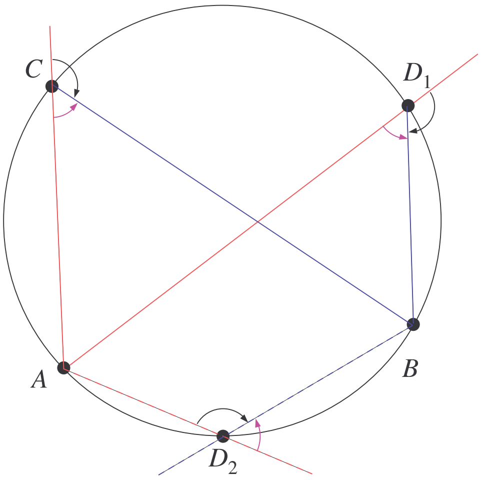

Figure 41: Cocyclicity and angles

Unformal proof. Look at the Figure 42 and remember that OAM and OBM are isosceles triangles. ■

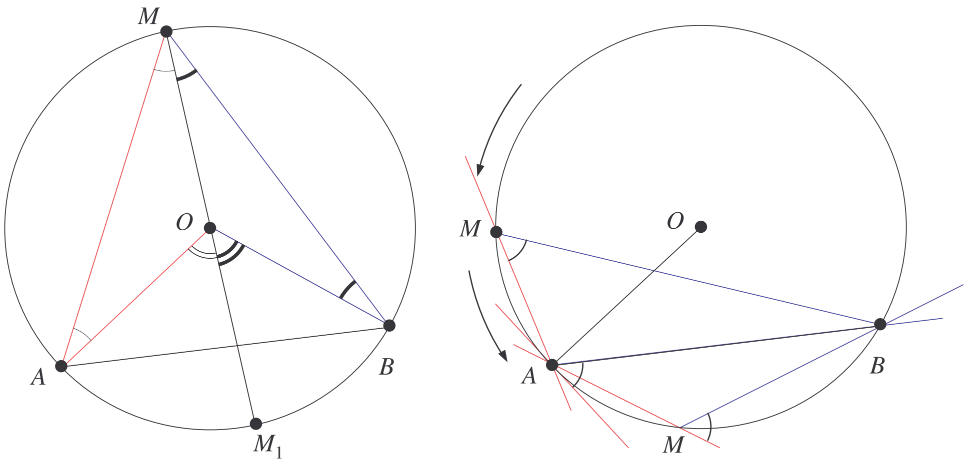

Figure 42: Cocyclicity and angles (unformal proof)

Exercise 5.1.5 Let A and B be two points and α an oriented angle of lines. Let Γ be the circle which is the set of points M such that $\langle\overleftrightarrow{MA}, \overleftrightarrow{MB}\rangle = \alpha$. Determine the tangent to Γ in A and the tangent to Γ in B. Which classical theorem do you get when $\alpha = \delta$ (where δ was defined in Exercise 5.1.2)?

Exercise 5.1.6 Let c and d be two lines secant in a point S. Let A and B be two points. Using the result of preceding exercise construct the circle Γ which is the set of points M such that $\langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle = \langle c, d \rangle$.

Exercise 5.1.7 If we choose an orthonormal frame to the plane, we have a bijection of the plane on the set of complex numbers $\mathbb{C}$, the image of a point M denoted z_M is called the *affix* of M. In such a way we can consider the real plane as a complex line. Show that the points A, B, C and D are cocyclical if and only if the cross-ratio of their affixes $[z_A, z_B; z_C, z_D]$ is a real number.

Proposition 5.1.8 If four points A, B, C and D are on a circle then the bisectors of the couples of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, and $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are parallel.

You can follow the proof without looking at the picture!

Proof. Let d be a bisector of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, that means that $\langle \overleftrightarrow{AB}, d \rangle = \langle d, \overleftrightarrow{CD} \rangle$ and so $2\langle \overleftrightarrow{AB}, d \rangle = \langle \overleftrightarrow{AB}, \overleftrightarrow{CD} \rangle$. Let d' be a bisector of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, we get: $\langle \overleftrightarrow{BD}, d' \rangle = \langle d', \overleftrightarrow{AC} \rangle$ and so $2\langle d', \overleftrightarrow{AC} \rangle = \langle \overleftrightarrow{BD}, \overleftrightarrow{AC} \rangle$. By a direct application of the rules we get:

$$\langle d', d \rangle = \langle d', \overleftrightarrow{AC} \rangle + \langle \overleftrightarrow{AC}, \overleftrightarrow{AB} \rangle + \langle \overleftrightarrow{AB}, d \rangle.$$

Taking this equality twice:

$$2\langle d', d \rangle = 2\langle d', \overleftrightarrow{AC} \rangle + 2\langle \overleftrightarrow{AC}, \overleftrightarrow{AB} \rangle + 2\langle \overleftrightarrow{AB}, d \rangle = \langle \overleftrightarrow{BD}, \overleftrightarrow{AC} \rangle + 2\langle \overleftrightarrow{AC}, \overleftrightarrow{AB} \rangle + \langle \overleftrightarrow{AB}, \overleftrightarrow{CD} \rangle$$

So:

$$2\langle d', d \rangle = \langle \overleftrightarrow{BD}, \overleftrightarrow{AB} \rangle + \langle \overleftrightarrow{AC}, \overleftrightarrow{CD} \rangle$$

or

$$2\langle d', d \rangle = \langle \overleftrightarrow{CA}, \overleftrightarrow{CD} \rangle - \langle \overleftrightarrow{BA}, \overleftrightarrow{BD} \rangle.$$

If our four points are cocyclical, then the second member of this equality is zero, and so $\langle d', d \rangle = 0$ or δ. This means that $d \parallel d'$ or $d \perp d'$. In both cases the other bisectors of these two angles d_1 and d'_1 will also verify $d_1 \parallel d'$ or $d_1 \perp d'$, $d \parallel d'_1$ or $d \perp d'_1$, $d_1 \parallel d'_1$ or $d_1 \perp d'_1$. All these couples of relations are equivalent since $d \perp d_1$ and $d' \perp d'_1$. They all mean that the bisectors of the angles are parallel.

Now you can look at the picture, it helps to guess the shortest way to prove a result, but the formalism of OAL helps even more. ■

Exercise 5.1.9 Show the converse of preceding theorem, that is: let a, b, c and d be the four sides of a quadrilateral and let A, B, C, D, E and F be the vertices. The names are chosen in such a way that E and F do not belong to a common side of the quadrilateral. If the bisectors of the sides meeting in E are parallel to the bisectors of the sides meeting in F, then A, B, C and D are cocyclical (see Figure 43).

Cocyclicity and angle bisectors (link to JavaSketchpad animation)
`http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/CocyclicAngleBisectors.htm`

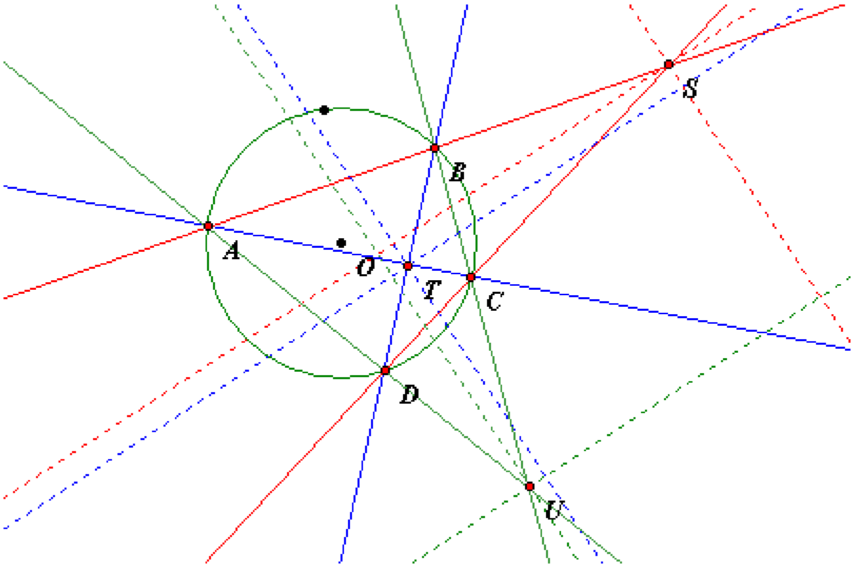

Figure 43: Cocyclicity and angle bisectors

Exercise 5.1.10 7. Let A, B, C, D, A', B', C' and D' be points such that there is one circle going through A, B, A' and B', one circle going through B, C, B' and C', one circle going through C, D, C' and D', one circle going through D, A, D' and A'. Show that A', B', C' and D' are cocyclical if and only if A, B, C and D are cocyclical (see Figure 44).

Cocyclicity and four circles (link to JavaSketchpad animation)
`http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/CocyclicityAndFourCircles.htm`

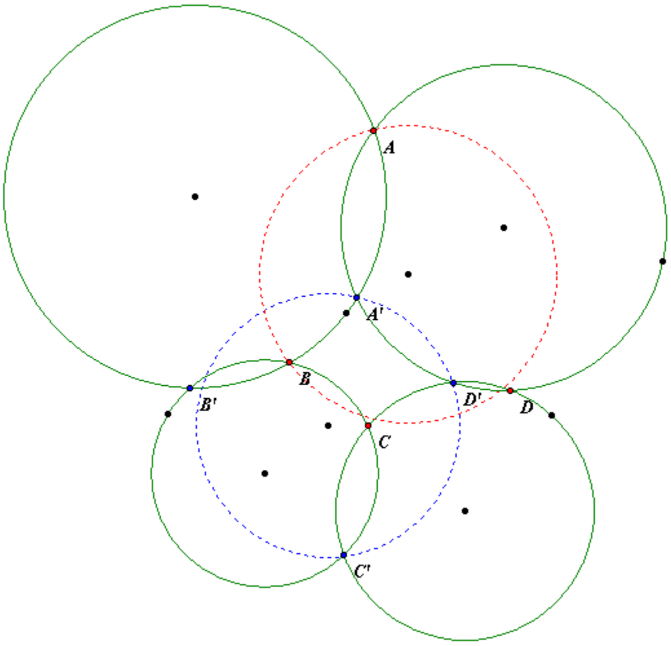

Figure 44: Cocyclicity and four circles

Proposition 5.1.11 Let A and B be two points. The set of points M such that $\overleftrightarrow{MA} \perp \overleftrightarrow{MB}$ is the circle with diameter AB.

Proof. In a rectangular triangle the median is equal to half the hypothenuse, see Figure 45. ■

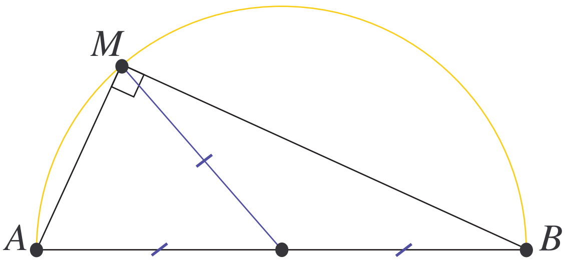

Figure 45: Rectangular triangle and circle

Exercise 5.1.12 Let ABC be a triangle and let M be a point. Call P, Q and R the orthogonal projections of M on the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ of the triangle. Find the set of points such that P, Q and R are collinear. The line on which these points are is called *the Simson line* of the point M relatively to the triangle ABC.

Exercise 5.1.13 Let $ABCH$ be an orthocentric quadrangle. We call A', B', C', A'', B'' and C'' the midpoints of BC, CA, AB, HA, HB and HC. Show that $A'A''$, $B'B''$ and $C'C''$ are diameters of the common Euler circle to ABC, ABH, AHC and HBC.

5.2 A peculiar multiplication by 2

5.2.1 Oriented angles of rays

As for the oriented angles of lines, we will postpone the definition of oriented angles of rays to a later part of this chapter, but we will give now the rules they follow. Two rays with common origin $\overrightarrow{SA}$ and $\overrightarrow{SB}$ determine an oriented angle of rays beginning at $\overrightarrow{SA}$ and ending at $\overrightarrow{SB}$. We denote that oriented angle of rays by $(\overrightarrow{SA}, \overrightarrow{SB})$. We want:

$$\begin{array}{rcl}
(\overrightarrow{SA}, \overrightarrow{SB}) & = & (\overrightarrow{SA}, \overrightarrow{SC}) + (\overrightarrow{SC}, \overrightarrow{SB}) \\
(\overrightarrow{SB}, \overrightarrow{SA}) & = & -(\overrightarrow{SA}, \overrightarrow{SB}) \\
(\overrightarrow{SA}, \overrightarrow{SA}) & = & 0 \\
(\overrightarrow{f(S)f(A)}, \overrightarrow{f(S)f(B)}) & = & (\overrightarrow{SA}, \overrightarrow{SB}) \text{ for any direct similarity } f \\
(\overrightarrow{g(S)g(A)}, \overrightarrow{g(S)g(B)}) & = & -(\overrightarrow{SA}, \overrightarrow{SB}) \text{ for any indirect similarity } g
\end{array}$$

We will denote by OAR the set of oriented angles of rays.

Exercise 5.2.1 Let $\overrightarrow{SA}$, $\overrightarrow{SB}$, $\overrightarrow{SC}$ and $\overrightarrow{SD}$ be four rays with common origin S. Show that:

$$(\overrightarrow{SA}, \overrightarrow{SB}) = (\overrightarrow{SC}, \overrightarrow{SD}) \text{ if and only if } (\overrightarrow{SA}, \overrightarrow{SC}) = (\overrightarrow{SB}, \overrightarrow{SD}).$$

As for angles of lines, we can study the equation $2x = \alpha$ in OAR, for a given α in OAR. The equation $2x = 0$ has two solutions: 0 and one which is different from zero and which we call a *flat angle* and denote by p. Then the equation $2x = \alpha$ in x in OAR has two solutions whose difference is p. An angle of rays, in the meaning of a couple of rays with common origin, will have two rays bisecting it, the union of these two rays is a line which is called the *bisector of the angle of rays* (see Figure 46).

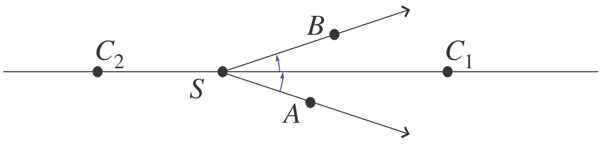

Figure 46: Bisector of angle of rays

5.2.2 Maps between OAL and OAR

Given an oriented angle of rays α, we can represent it through two rays with common origin like $\overrightarrow{SA}$ and $\overrightarrow{SB}$. If we consider the lines $\overleftrightarrow{SA}$ and $\overleftrightarrow{SB}$ we can define an oriented angle of lines $\langle\overleftrightarrow{SA}, \overleftrightarrow{SB}\rangle$. We may call it $\overleftrightarrow{\alpha}$, because this oriented angle of lines is independent of the choice of the rays $\overrightarrow{SA}$ and $\overrightarrow{SB}$ such that $(\overrightarrow{SA}, \overrightarrow{SB}) = \alpha$, defining a map from OAR in OAL, see Figure 47. This map is surjective from OAR onto OAL, but it is not injective. In fact α and $\alpha + p$ have the same image.

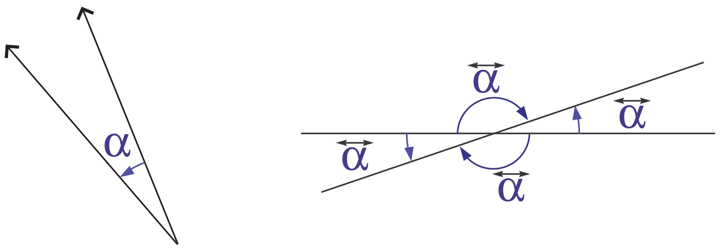

Figure 47: Oriented angles of rays and lines

We have another map from OAR on OAL. Given α in OAR, we can choose $\overrightarrow{SA}$ and $\overrightarrow{SB}$ such that $(\overrightarrow{SA}, \overrightarrow{SB}) = \alpha$. Let M_1M_2 be the bisector of the angle of rays $(\overrightarrow{SA}, \overrightarrow{SB})$ in the meaning of a couple of rays with common origin. We get an oriented angle of lines $(\overleftrightarrow{SA}, \overleftrightarrow{M_1M_2})$ which is independent of the choice of $\overrightarrow{SA}$ and $\overrightarrow{SB}$. We will denote it by a symbol depending only on α, like $\frac{1}{2} * \alpha$ for instance. The map OAR $\rightarrow$ OAL, $\alpha \mapsto \frac{1}{2} * \alpha$ is a bijection. We will denote the inverse map by OAL $\rightarrow$ OAR, $x \mapsto 2 * x$. Do not make the confusion between $2 * x$ and $2x$. The quantity $2 * x$ belongs to OAR but $2x$ belongs to OAL.

Exercise 5.2.2 If $x \in$ OAL, is it true that $\overleftrightarrow{2 * x} = 2x$? Is it true that $\overleftrightarrow{2x} = 2 * x$?

The relation between $x = \frac{1}{2}\alpha$ in OAL and $2 * x = \alpha$ in OAR can be illustrated by the Figure 48.

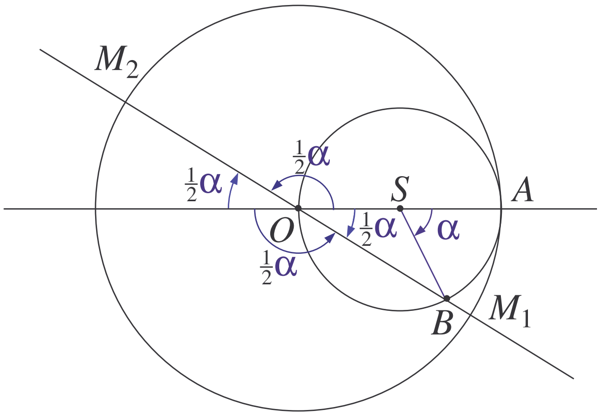

Figure 48: Oriented angle and half angle

5.2.3 Rotating mirror

We consider a plane mirror rotating around a line lying in the plane of the mirror and a light ray falling on the mirror along a direction orthogonal to the axis of rotation of the mirror. All this means that we can draw our picture in the plane orthogonal to the axis of rotation and containing the light ray, see Figure 49. We suppose the mirror is first in a position which intersects the plane of the figure through the line m_1 and then in a position which intersects the plane of the figure through the line m_2. The change can be described by the oriented angle of lines $\langle m_1, m_2\rangle$. What is going to happen to the reflected light ray?

Let us call R_1 the point where the light ray hits m_1 and R_2 the point where the light ray hits m_2. Choose a point A_0 on the incoming ray, A_1 on the first outgoing ray and A_2 on the second.

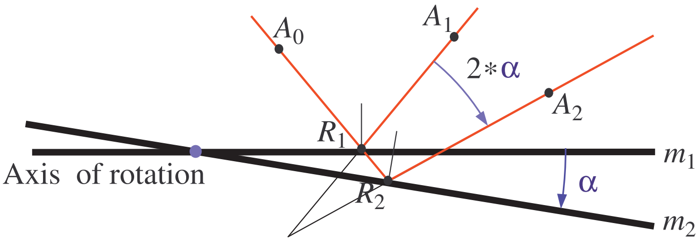

Figure 49: Rotating mirror

The law of reflection of light tells us that

$$\begin{aligned}
(\overrightarrow{R_1A_0}, \overrightarrow{R_1A_1}) &= 2 * \left[\langle \overleftrightarrow{R_1A_0}, m_1\rangle + \delta\right] \\
(\overrightarrow{R_2A_0}, \overrightarrow{R_2A_2}) &= 2 * \left[\langle \overleftrightarrow{R_2A_0}, m_2\rangle + \delta\right]
\end{aligned}$$

So that

$$(\overrightarrow{R_1A_1}, \overrightarrow{R_2A_2}) = 2 * \langle m_1, m_2\rangle.$$

5.2.4 Back to cocyclicity

Theorem 5.2.3 Let O be a point at equal distances from two points A and B. A point M belongs to a circle of center O and going through A and B if and only if:

$$(\overrightarrow{OA}, \overrightarrow{OB}) = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle.$$

Proof. 1. Suppose M belongs to the circle of center O and going through A and B. Denote by M_1 the point of the circle collinear with O and M (see Figure 50).

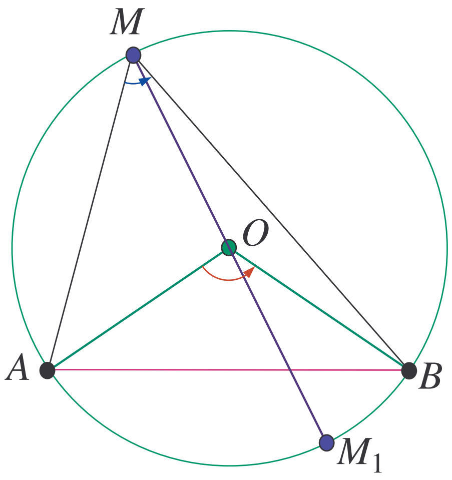

Figure 50: Angles corresponding to the center and a point on the circle

We have

$$(\overrightarrow{OA}, \overrightarrow{OM}) = (\overrightarrow{OA}, \overrightarrow{OM_1}) + (\overrightarrow{OM_1}, \overrightarrow{OM}) = (\overrightarrow{OA}, \overrightarrow{OM_1}) + p.$$

But $(\overrightarrow{OA}, \overrightarrow{OM_1}) = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MM_1} \rangle$. In the same way $(\overrightarrow{OM}, \overrightarrow{OB}) = 2 * \langle \overleftrightarrow{MM_1}, \overleftrightarrow{MB} \rangle + p$.

So since

$$(\overrightarrow{OA}, \overrightarrow{OB}) = (\overrightarrow{OA}, \overrightarrow{OM}) + (\overrightarrow{OM}, \overrightarrow{OB})$$

and since $p + p = 0$, we get :

$$(\overrightarrow{OA}, \overrightarrow{OB}) = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MM_1} \rangle + 2 * \langle \overleftrightarrow{MM_1}, \overleftrightarrow{MB} \rangle,$$

or finally $(\overrightarrow{OA}, \overrightarrow{OB}) = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle$.

2. Suppose $(\overrightarrow{OA}, \overrightarrow{OB}) = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle$.

Let us call $\alpha := (\overrightarrow{OA}, \overrightarrow{OB})$. The circle with center O and going through A and B will intersect the line AM in A and in an other point, let us call it N. As a consequence of the first part of the proof

$$2 * \langle \overleftrightarrow{NA}, \overleftrightarrow{NB} \rangle = \alpha = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle.$$

But $x \mapsto 2 * x$ is bijective so $\langle \overleftrightarrow{NA}, \overleftrightarrow{NB} \rangle = \langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle$.

Since $\overleftrightarrow{NA} = \overleftrightarrow{MA}$, that means that $\overleftrightarrow{NB}$ and $\overleftrightarrow{MB}$ are parallel. Having the point B in common these two lines are the same line and $M = N$. ■

5.3 Different types of angles

5.3.1 The angles as subsets of the plane

.

A triangle ABC can be viewed as the convex hull of its vertices, that means as the set of points with only positive or null normed barycentric coordinates relatively to the three points A, B and C. We can also consider this set as the intersection of closed half-planes.

In the same way an angle can be considered as the intersection of two half-planes …but of course sometimes you get a strip and sometimes nothing at all.

You can also consider the open set topological interior of the preceding one.

Sometimes you call also angle the part of the plane complementary to the preceding ones, see Figure 51.

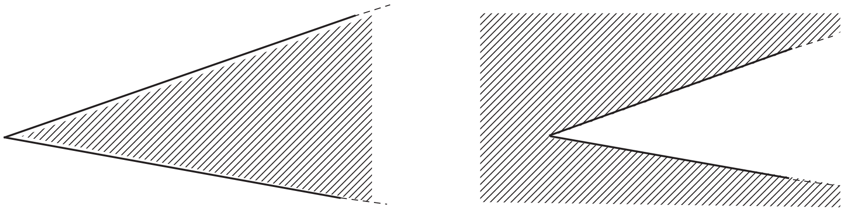

Figure 51: Angles as intersections of half-planes

On these types of angles you do not define any addition, the relevant operations are the union and the intersection.

These angles have very poor properties, even if they can be very useful, for instance in complex analysis. We won't talk about them any more because our aim is to understand much more abstract concepts which are also called angles.

5.3.2 Oriented or unoriented angles of lines or rays

The title of this paragraph contains 4 types of angles. We will characterize each type by the solution of following problem:

> Given two points A and B and an angle α determine the locus of the points M viewing AB under the angle α.

1. If α is an **oriented angle of lines** the locus of the points M such that the angle of the **couple** of lines $\overleftrightarrow{MA}$ and $\overleftrightarrow{MB}$ is equal to α is a circle going through A and B.

2. If α is an **oriented angle of rays** the locus of the points M such that the angle of the **couple** of rays $\overrightarrow{MA}$ and $\overrightarrow{MB}$ is equal to α is an arc of circle with extremities A and B.

3: If α is an **unoriented angle of lines** the locus of the points M such that the angle of the **pair** of lines $\overleftrightarrow{MA}$ and $\overleftrightarrow{MB}$ is equal to α is the union of two circles going through A and B, symmetrical in the reflection in the line $\overleftrightarrow{AB}$.

4. If α is an **unoriented angle of rays** the locus of the points M such that the angle of the **pair** of rays $\overrightarrow{MA}$ and $\overrightarrow{MB}$ is equal to α is the union of two arcs of circles with extremities A and B, symmetrical in the reflection in the line $\overleftrightarrow{AB}$.

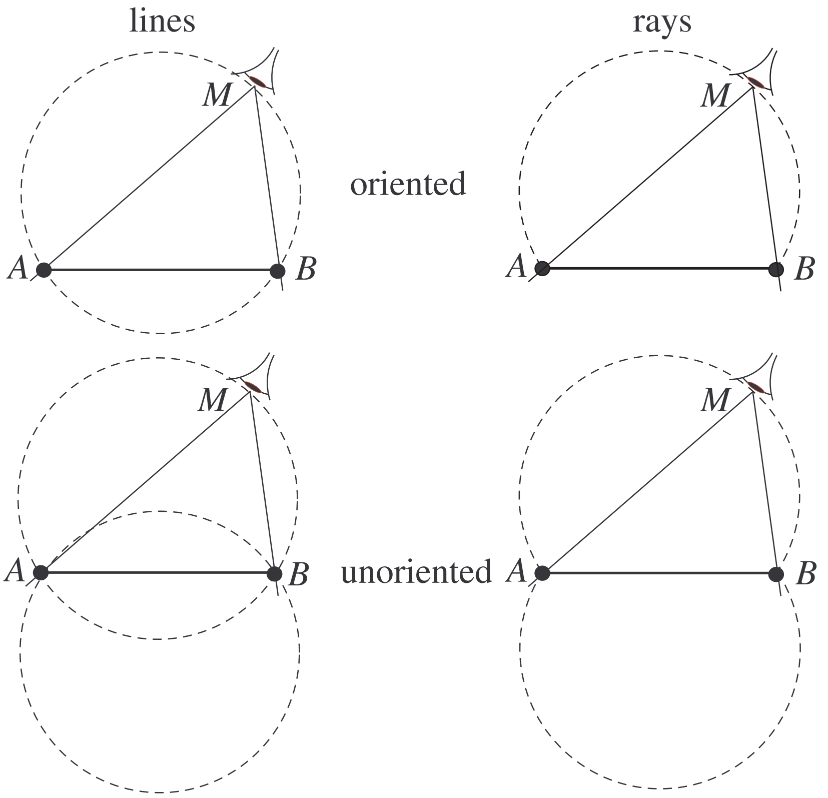

Figure 52: Segment viewed from different angle types

Exercise 5.3.1 Let PQR be an equilateral triangle, with center G. Let A and B be two distinct points. Draw the locus of the points M viewing AB under the same angle as G views QR, when the word angle means successively the four meanings described above.

5.4 Definition of angles

5.4.1 The groups of similarities and of direct similarities

The group of similarities of the plane, denoted **Sim**$(\mathcal{P})$ is isomorphic to the following group of real invertible matrices

$$G := \left\{ \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } \left\{ \begin{array}{rcl} a_{11}a_{22} - a_{12}a_{21} & \neq & 0 \\ (a_{11})^2 + (a_{12})^2 & = & (a_{21})^2 + (a_{22})^2 \\ a_{11}a_{21} + a_{12}a_{22} & = & 0 \end{array} \right. \right\}$$

A simple calculation shows that G is the disjoint union of G^+ and G^-, where

$$G^+ = \left\{ \begin{bmatrix} \alpha & \beta & b_1 \\ -\beta & \alpha & b_2 \\ 0 & 0 & 1 \end{bmatrix} \middle| \, \alpha^2 + \beta^2 \neq 0 \right\} \text{ and } G^- = \left\{ \begin{bmatrix} \alpha & \beta & b_1 \\ \beta & -\alpha & b_2 \\ 0 & 0 & 1 \end{bmatrix} \middle| \, \alpha^2 + \beta^2 \neq 0 \right\}.$$

Check that G^+ is a subgroup of G, but that G^- is not. We call **Sim**$^+(\mathcal{P})$ the subgroup of **Sim**$(\mathcal{P})$ isomorphic to G^+.

You can also consider the map **Sim**$(\mathcal{P}) \to \mathbb{R}^*$, $t \mapsto \Delta(t)$, where $\Delta(t) := a_{11}a_{22} - a_{12}a_{21}$, this quantity being independent of the frame. Then **Sim**$^+(\mathcal{P}) = \Delta^{-1}(]0, +\infty[)$.

5.4.2 Definitions of the sets of angles

The groups **Sim**$(\mathcal{P})$ and **Sim**$^+(\mathcal{P})$ are acting on the set $\mathcal{D} \times \mathcal{D}$. The orbits of **Sim**$(\mathcal{P})$ are the unoriented angles of lines, the orbits of **Sim**$^+(\mathcal{P})$ are the oriented angles of lines.

Another way to say the same thing is to define an equivalence relation in $\mathcal{D} \times \mathcal{D}$: two couples of lines (a, b) and (c, d) are called *equivalent* (respectively plus-equivalent) if there is a (direct) similarity t such that $t(a) = c$ and $t(b) = d$. Let us call this relation $\mathcal{R}$ (respectively $\mathcal{R}^+$). You have to verify that $\mathcal{R}$ (respectively $\mathcal{R}^+$) is an equivalence relation. In fact this follows easily from the fact that the (direct) similarities form a group. Then you define $\mathrm{OAL} := (\mathcal{D}\times\mathcal{D})/\mathcal{R}^+$ and the set of unoriented angles of lines by $\mathrm{UAL} := (\mathcal{D} \times \mathcal{D})/\mathcal{R}$.

Exercise 5.4.1 Explain how to define the oriented and unoriented angles of rays.

5.4.3 Definitions of the addition

Once, the sets are defined you have to define an operation on these sets to get groups. This is done only for the set of oriented angles.

Let us begin with OAR. First you show that if a and b are rays with same origin, and if t is a direct similarity, then $(t(a), t(b)) = (a, b)$ if and only if $(a, t(a)) = (b, t(b))$. So we can associate an oriented angle of rays to each direct similarity. Then we will define for angles expressed all with rays of same origin:

> $(a, b) + (c, d) = (e, f)$ iff there exist two direct similarities t_1 and t_2 such that $b = t_1(a)$, $d = t_2(c)$ and $f = (t_2 \circ t_1)(e)$.

To define the addition on OAL you can use the bijection $\frac{1}{2}*$ and $2*$.

5.5 Measure of angles

To do all what has been done until now, you did not need an oriented plane and you did not need to know trigonometry or anything about π.

5.5.1 The groups $\mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{R}/\pi\mathbb{Z}$

Definition 5.5.1 $\mathbb{R}/2\pi\mathbb{Z}$ is the set of subsets of $\mathbb{R}$ which are of the form $\{\, x + 2k\pi \mid k \in \mathbb{Z} \,\}$, where $x \in \mathbb{R}$.

Exercise 5.5.2 Define an addition on $\mathbb{R}/2\pi\mathbb{Z}$. Show why it is impossible to define a multiplication $\cdot$ on that set which would verify:

$$\{\, x + 2k\pi \mid k \in \mathbb{Z} \,\} \cdot \{\, y + 2h\pi \mid h \in \mathbb{Z} \,\} = \{\, xy + 2m\pi \mid m \in \mathbb{Z} \,\}$$

for any x and y in $\mathbb{R}$.

Exercise 5.5.3 Define $\mathbb{R}/\pi\mathbb{Z}$, and an addition on $\mathbb{R}/\pi\mathbb{Z}$. Define what $2\{\, x + k\pi \mid k \in \mathbb{Z} \,\}$ means: We put

$$2 * \{\, x + k\pi \mid k \in \mathbb{Z} \,\} = \{\, 2x + 2k\pi \mid k \in \mathbb{Z} \,\} \in \mathbb{R}/2\pi\mathbb{Z}.$$

Define a group isomorphism between $\mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{R}/\pi\mathbb{Z}$.

5.5.2 Measures

Theorem 5.5.4 There are two group homomorphisms of OAR on $\mathbb{R}/2\pi\mathbb{Z}$.

This theorem is difficult to prove. The simplest way is to show explicitly an isomorphism of OAR on the set of complex numbers of module 1, which is a group as subgroup of $\mathbb{C}^*$. Then for any real number θ you define $\sum_{k=0}^{\infty} \frac{1}{k!}(i\theta)^k$. You show that this function is periodical and define the period as 2π.

To choose one of the two homomorphisms is a way of choosing an orientation for $\mathcal{P}$. The usual way to orient $\mathcal{P}$ is to decide which frames are going to be called direct, the other becoming indirect. Once this choice is done you have only one homomorphism of OAR on $\mathbb{R}/2\pi\mathbb{Z}$, and only one of OAL on $\mathbb{R}/\pi\mathbb{Z}$.

Often the measures of angles are written as the angles and since equality in $\mathbb{R}/2\pi\mathbb{Z}$ is written $\ldots = \ldots \quad (2\pi)$, the fundamental theorem saying that a point M belong to the circle of center and going through A and B if and only if $(\overrightarrow{OA}, \overrightarrow{OB}) = 2 * \langle \overleftrightarrow{MA}, \overleftrightarrow{MB} \rangle$ will be written:

$$(\overrightarrow{OA}, \overrightarrow{OB}) = 2(\overrightarrow{MA}, \overrightarrow{MB}) \quad (2\pi).$$

But that is a way to mess up all the structures. So you better know what you mean before you do that!

6 A survey of the history of Geometry

6.1 Ancient Greek geometry

6.1.1 Before Euclid

Egypt, Babylonia, India and China

- Usual computation formulae for areas
- Approximations of π: ≈ 3, or in Egypt $\frac{256}{81} = 3{,}1604\ldots$, in India $3{,}1088$.

  Also: Area $=$ Circumference$\times$ Diameter$/4$.
- Pythagorean triplets. Definition: (a, b, c) in $\mathbb{N}^3$ such that: $a^2 + b^2 = c^2$.

  1700 years before Christ in Babylon: (12 709, 13 500, 18 541).

Thales ($\approx$ 624-547)

- from Miletus in Asia Minor
- first mentioned Greek mathematician
- credited with beginning the Greek mathematical tradition

In France, his name is given to the fundamental theorem that says that parallel projection of a line on an other preserves the affine structure of lines in an affine plane or space.

Pythagoras ($\approx$ 572-497)

"Number is the substance of all things". Numbers means for him and his school: one, two, three, …

Figure 53: Numbers and relations of segments

For example the two segments in Figure 53 are as 5 to 7. These theories where closely related to music. Most of the early words in Greek mathematics come from music.

They discovered parity and the rules of addition and multiplication modulo 2:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Applying these rules you see that you can always choose p and q of different parity when you put a statement of the type: "this magnitude is to that magnitude as p is q".

Now, consider a triangle which is at the same time rectangle and isosceles. The hypotenuse is to the side as p to q. From the theorem of Pythagoras we get: $p^2 = 2q^2$. We deduce that p is even and can thus be written $p = 2k$, and so $4k^2 = 2q^2$ or $q^2 = 2k^2$, which shows that q is also even. CONTRADICTION !

The diagonal and the side of a square are ”incommensurable”. In modern language we would say that they proved the irrationality of $\sqrt{2}$.

Three problems were raised at that time:

Problem. 1. To square a circle: given a circle C of radius R, find a length a such that the square constructed on a has the same area as the disc limited by the circle C.

Problem. 2. To duplicate a cube: Given a cube Γ with edge a, construct with compass and ruler a length b such that the volume of the cube with edge b is twice the volume of Γ.

Problem. 3. To trisect an angle.

Plato (429-347)

- organized seminar in mathematics in the Academy, name of the place where he was teaching.
- knew well and gave great importance to the five regular polyhedra:

name	Faces	Edges	Vertices	Shape of faces
Tetrahedron	4	6	4	triangles
Cube	6	12	8	squares
Octahedron	8	12	6	triangles
Dodecahedron	12	30	20	pentagons
Icosahedron	20	30	12	triangles

In modern language these are figures invariant by the finite subgroups of the group of isometries with a fixed point or the finite groups of isometries of a three-dimensional euclidean vector space which are not isomorphic to subgroups of isometries of the plane.

The regular polyhedra were thought as the constitutive elements of: fire, earth, air, water and ETHER.

We will encounter the same concepts of ether and groups in the foundations of modern physics.

Aristoteles (384-322)

The place where Aristoteles was teaching was the Lyceo.

One of his students was Alexander the great, who conquered Egypt, founded Alexandria and died in 323, giving a Greek king to Egypt: Ptolemy I.

Ptolemy I founded the Museum, temple of the muses and the Library.

He called for all the best scholars in the Greek world and got Euclid, probably among many others.

6.2 Euclid

Only two stories are told about him:

1. To Ptolemy who was asking if it was necessary to go through all the propositions to learn geometry, he answered that ”there is no royal path to geometry”.

2. When a student asked him what he would earn by learning geometry, he asked his slave: ”give that man some piece of money since he needs to earn something for everything he does”.

Euclid wrote several works. Most of them are lost, but the best known work, called ”The Elements”, has been copied on and on. You have a very nice presentation and translation on the net:

`http://aleph0.clarku.edu/\symbol{126}djoyce/java/elements/elements.html`

The Elements can be considered as:

- the foundation of geometry
- the foundation of mathematics as a deductive activity
- containing a theory equivalent to the theory of real numbers.

The Elements is composed of 13 books. Book I is devoted to the proof of the theorem of Pythagoras, the last proves that there are no regular polyhedra but the five Platonic bodies.

Euclid is building his theory on definitions, on axioms, which are logical common truths, and on 5 postulates which are rules you have to admit at the start: The postulates 1, 2 and 3 give the possibility to draw a line between two points, to extend a line and to draw a circle with given center through a given point. Postulate 4 says that all right angles are equal. The Postulate 5:

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The use of this postulate is postponed until proposition 29 !

The content of proposition 36: ”Parallelograms which are on equal bases and in the same parallels equal one another” would be stated in modern language:

$$\det(\overrightarrow{u}, \overrightarrow{v}) = \det(\overrightarrow{u}, \overrightarrow{v} + \lambda\overrightarrow{u}),$$

which is the main idea for bilinearity.

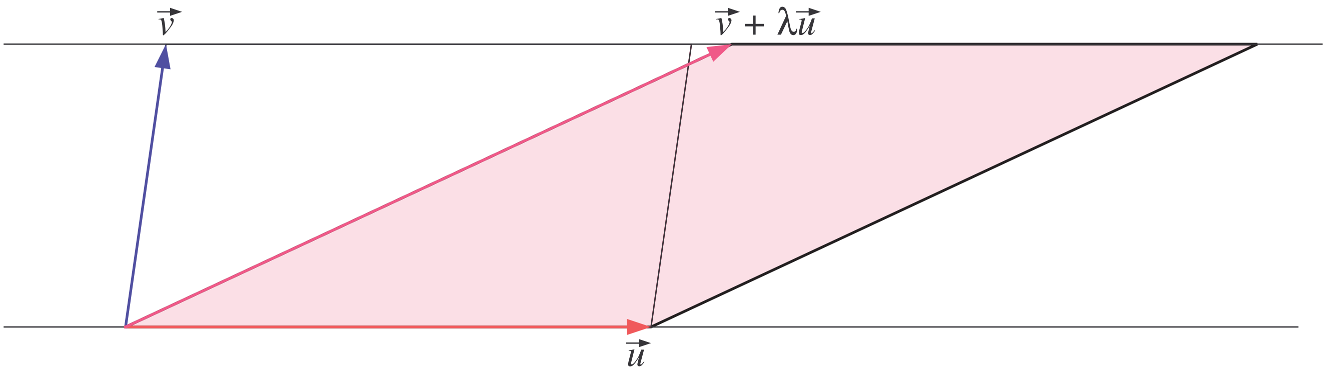

Figure 54: Parallelograms of equal base

The basic ideas of the proof of the theorem of Pythagoras are on Figure 55.

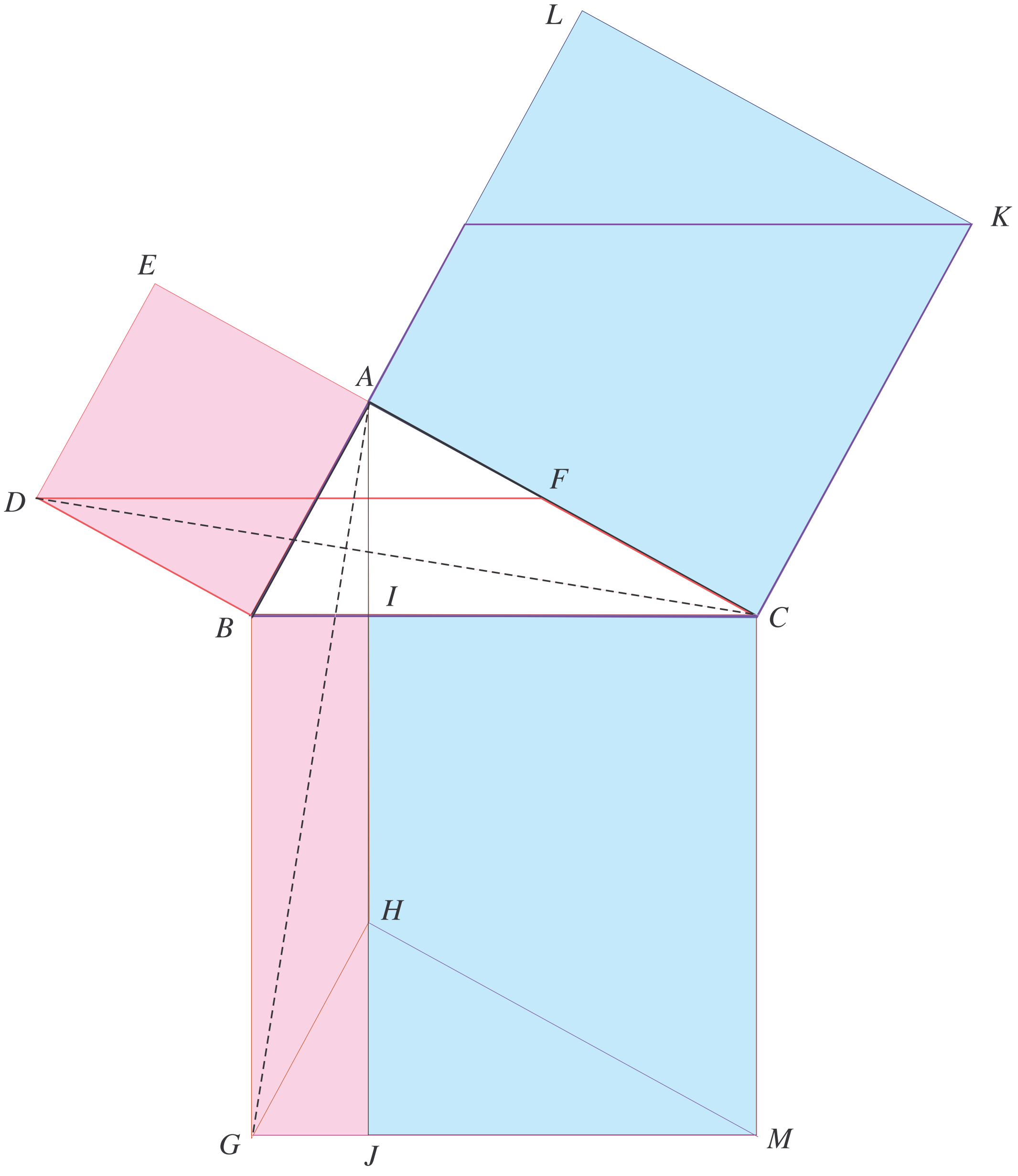

Figure 55: Proof of Pythagoras's theorem

$$\begin{aligned}
\text{Area}\, BDEA &= \text{Area}\, BDFC \\
&= 2\,\text{Area}\, DBC \\
&= 2\,\text{Area}\, ABG \\
&= \text{Area}\, ABGH \\
&= \text{Area}\, IBGJ
\end{aligned}$$

Similarly Area $ACKL =$ Area $IJMC$.

6.3 After Euclid

Archimedes of Syracuse ($\approx$ 287-212 B.C.E.)

Example: The area of a sphere is equal to the area of the circumscribed cylinder.

Apollonius of Perge ($\approx$ 250-175 B.C.E.)

Theory of conics: sections of a cone, tangents and normals, foci …

Claudius Ptolemy ($\approx$ 100-178)

The Almagest

Theorem 6.3.1 (Ptolemy's theorem) A convex quadrangle $ABCD$ is inscribed in a circle if and only if:

$$AC \times BD = AB \times CD + AD \times BC.$$

Proof. (This proof is of course completely anachronical). Let us call a, b, c and d the affixes of the points A, B, C and D considered as points of the complex plane. The above relation may be written:

$$|c-a|\,|d-b| = |b-a|\,|d-c| + |d-a|\,|c-b|$$

or:

$$|ab+cd-ad-bc| = |bd+ac-ad-bc| + |ab+cd-bd-ac|\,.$$

This relation of type $|z_1+z_2| = |z_1|+|z_2|$ is true if and only if there is a positive real number λ such that:

$$ab+cd-bd-ac = \lambda(bd+ac-ad-bc)$$

or:

$$\frac{d-a}{b-a}\cdot\frac{b-c}{d-c} \in \mathbb{R}^*_- = \left]-\infty, 0\right[\,.$$

This relation expressed with angles gives us:

$$\arg\frac{d-a}{b-a} = \arg\frac{d-c}{b-c} + \pi$$

or:

$$\text{oriented angle of rays}(\overrightarrow{AB}, \overrightarrow{AD}) = \pi + \text{oriented angle of rays}(\overrightarrow{CB}, \overrightarrow{CD}),$$

which says that the points A and C are on the same circle through B and D but on different arcs delimited by B and D (see Figure 56). ■

Heron of Alexandria (first century)

Area of a triangle: $S = \sqrt{p(p-a)(p-b)(p-c)}$, where a, b and c are the lengths of the sides of the triangle and $p = \frac{1}{2}(a+b+c)$.

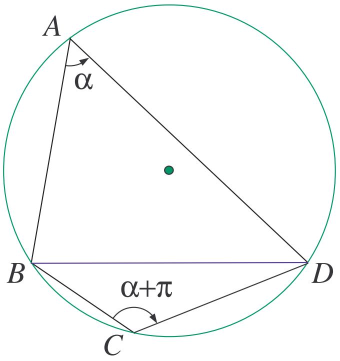

Figure 56: Convex quadrangle inside a circle

Menelaus (first century), Pappus (Alexandria, fourth century), Hypatia (≈ 355-415)

Theorem 6.3.2 (Pappus' theorem) Let A, B and C be three collinear points and A', B' and C' three points collinear on an other line. Let P be the intersection of BC' and $B'C$, Q the intersection of CA' and $C'A$ and R the intersection of AB' and $A'B$. Then the points P, Q and R are collinear (see Figure 57).

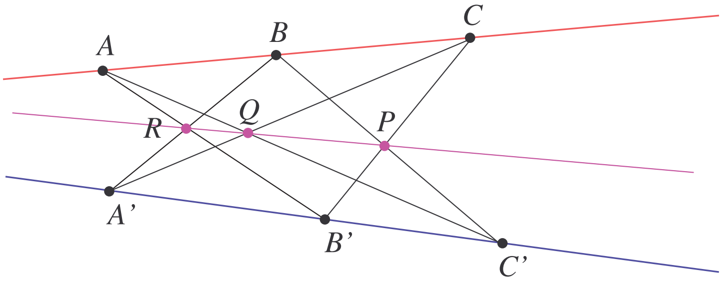

Figure 57: Pappus theorem about collinearity

6.4 From 800 to 1900

6.4.1 Transition

Islamic mathematics

al-Khwarizmi (≈ 780-850) in Bagdad

Omar Khayyam (1048-1131) in Isfahan

al-Kashi (early fifteenth century): computations with decimals. Nowadays, in France his name is given in secondary schools to the theorem that says that in a triangle:

$$a^2 = b^2 + c^2 - 2bc\cos\widehat{A}.$$

Medieval Europe

Leonardo of Pisa, called Fibonacci ($\approx$ 1170-1240).

Foundation of the university of Bologna, Paris and Oxford ($\approx$ 1200).

Distinction between potential infinity and actual infinity.

Renaissance

Perspective. Piero della Francesca (1420-1492)
Albrecht Dürrer (1471-1528)

From finite world to infinite universe

Galileo Galilei (1564-1642)

Johannes Kepler (1571-1630)

Isaac Newton (1642-1727)

6.4.2 Analytic geometry

Pierre de Fermat (1601-1665)

René Descartes (1596-1650)

6.4.3 Projective geometry

Girard Desargues (1591-1661)

Blaise Pascal (1623-1662)

Theorem 6.4.1 (Desargues' theorem) Two triangles are punctually perspective if and only if they are linely perspective.

Or:

Theorem 6.4.2 Let ABC and $A'B'C'$ be two triangles in a real plane Π: Let:

$$a := \overleftrightarrow{BC}, \quad b := \overleftrightarrow{CA}, \quad c := \overleftrightarrow{AB}, \quad a' := \overleftrightarrow{B'C'}, \quad b' := \overleftrightarrow{C'A'}, \quad c' := \overleftrightarrow{A'B'}.$$

Let P be the intersection of a and a', Q the intersection of b and b' and R the intersection of c and c'. Let $p := \overleftrightarrow{AA'}$, $q := \overleftrightarrow{BB'}$ and $r := \overleftrightarrow{CC'}$. The points P, Q and R are collinear if and only if p, q and r are concurrent.

Look at Figure 58 or the WWW-link

Desargues theorem for two triangles (3d) (link to JavaSketchpad animation)
`http://www.joensuu.fi/matematiikka/kurssit/TopicsInGeometry/TIGText/DesarguesTheoremForTwoTriangles3dJSP.htm`

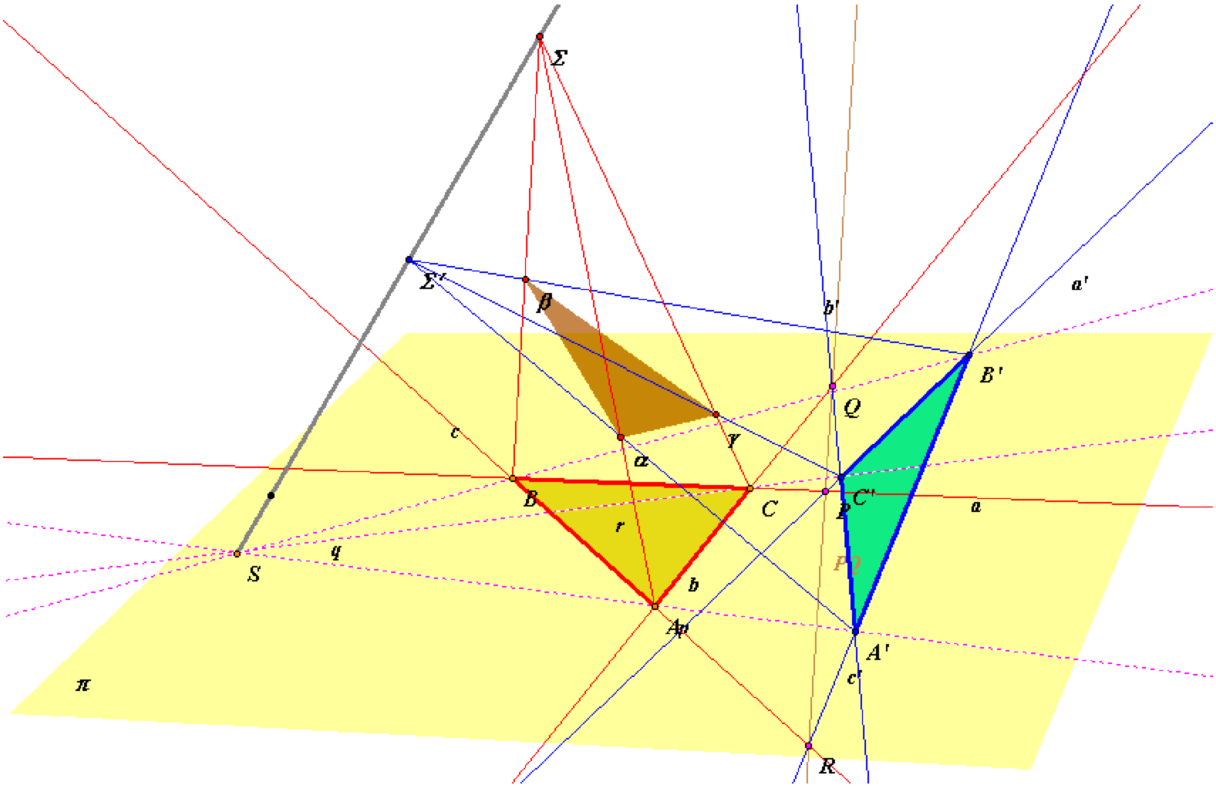

Figure 58: Desargues theorem for two triangles (3d)

Proof. Suppose that p, q and r are concurrent in a point S and let us show that P, Q and R are collinear.

Let d be a line going through S but not belonging to P and choose two points Σ and Σ' on d. The lines $\overleftrightarrow{\Sigma A}$ and $\overleftrightarrow{\Sigma' A'}$ are coplanar and thus intersecting in a point α (if the lines are parallel change the choice of Σ and/or Σ'). In the same manner $\overleftrightarrow{\Sigma B}$ and $\overleftrightarrow{\Sigma' B'}$ are intersecting in a point β and $\overleftrightarrow{\Sigma C}$ and $\overleftrightarrow{\Sigma' C'}$ are intersecting in a point γ. The lines $\overleftrightarrow{\alpha\beta}$ and $\overleftrightarrow{AB}$ are in the plane $A\Sigma B$, and thus intersecting in a point P_1 which belongs to the plane Π and to the plane $\alpha\beta\gamma$. The point P_1 is thus the intersection of $\overleftrightarrow{\alpha\beta}$ with Π. In the same way P_1 is the intersection of $\overleftrightarrow{\alpha\beta}$ with $\overleftrightarrow{A'B'}$. This point P_1 belonging to $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ is thus the point P. We have thus proved that P belongs to the plane $\alpha\beta\gamma$. In the same way, Q and R belong also to $\alpha\beta\gamma$. Thus the three points P, Q and R belong to the plane Π and to the plane $\alpha\beta\gamma$ and thus to their common line: the three points P, Q and R are collinear, see Figure 59. ■

Projective geometry in one dimension

Let a and a_1 be two lines, S and S_1 be two points which do not belong to these lines. We call *central projection* of a on a_1 with center S the map from a to a_1 such that the image of a point M of a is the point M_1 of a_1 collinear with S and M, see Figure 60.

Let us call M' the image of M_1 by the central projection of a_1 on a with center S_1. We have thus got a map from a to a, let us call it f. This map f is not a bijection: the intersection point N of a with the parallel to a_1 through S has no image and the intersection point M'_∞ of a with the parallel to a_1 through S_1 is not the image of any point. But if we add one abstract element

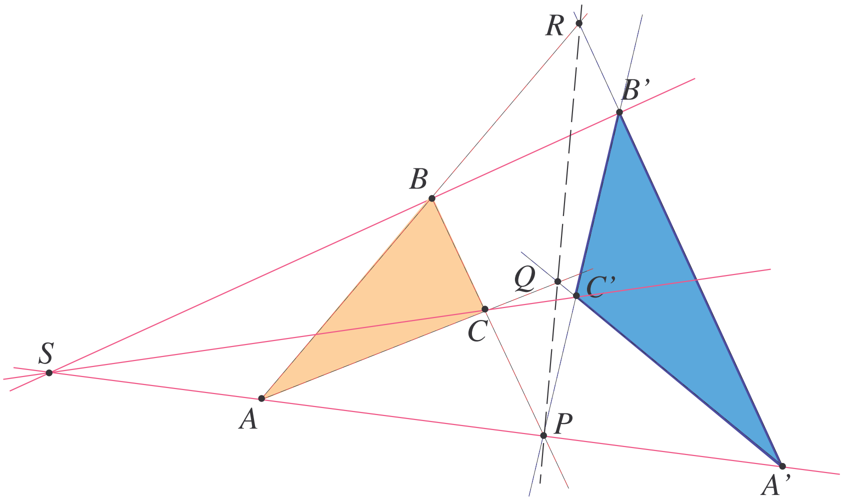

Figure 59: Desargues theorem for two triangles

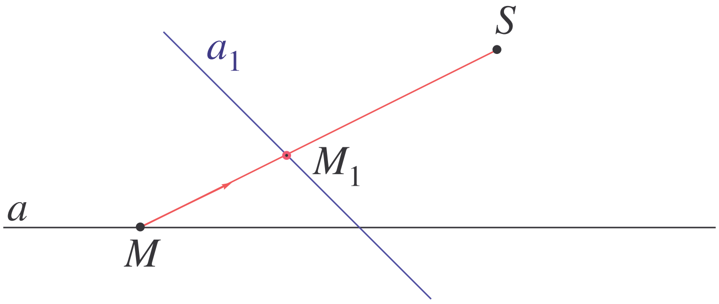

Figure 60: Central projection with center S

∞_a to the line a getting $\widetilde{a} = a \cup \{\infty_a\}$, and if we extend the definition of f by:

$$f(N) = \infty_a \qquad \text{and} \qquad f(\infty_a) = M'_\infty$$

then we have a bijection, see 61.

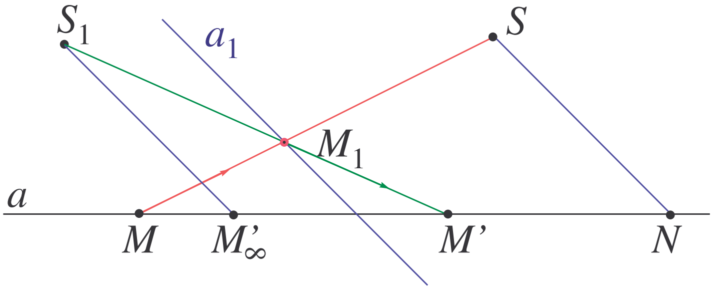

Figure 61: Central projection map from a to a

It is a graphic representation of a bijection of the circle and the line $\widetilde{a} = a \cup \{\infty\}$, see Figure 62.

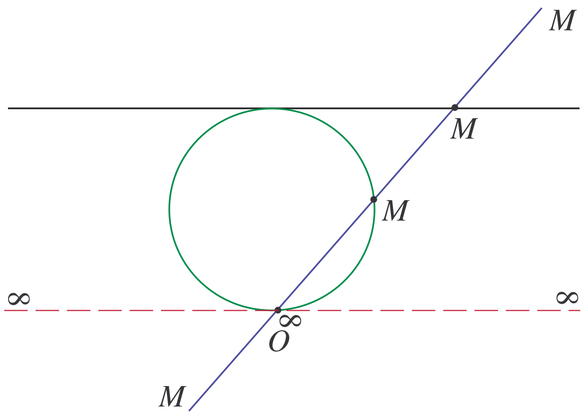

Figure 62: Threee graphic representations of the projective line

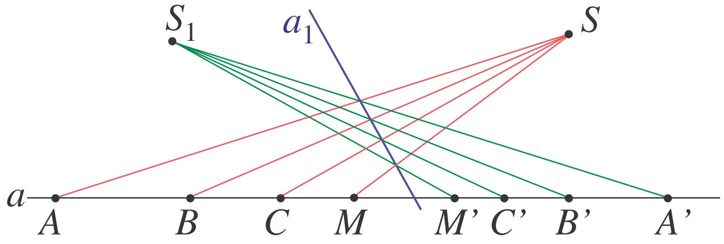

Figure 63: Central projection and cross-ratios

Both central projections are preserving cross-ratios, hence f preserves cross-ratios. Let A, B and C be three points of a and A', B' and C' their images by f, see Figure 63.

For any point M of a the image M' is thus such that:

$$[M', A'; B', C'] = [M, A; B, C].$$

If we take a frame on the line a, we get:

$$\frac{x_{B'} - f(x)}{x_{C'} - f(x)} \cdot \frac{x_{C'} - x_{A'}}{x_{B'} - x_{A'}} = \frac{x_B - x}{x_C - x} \cdot \frac{x_C - x_A}{x_B - x_A}$$

or

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

The functions of this form are called *homographic functions*. They form a group, since:

$$\frac{\alpha \frac{\alpha' x + \beta'}{\gamma' x + \delta'} + \beta}{\gamma \frac{\alpha' x + \beta'}{\gamma' x + \delta'} + \delta} = \frac{(\alpha\alpha' + \beta\gamma')x + (\alpha\beta' + \beta\delta')}{(\gamma\alpha' + \delta\gamma')x + (\gamma\beta' + \delta\delta')}.$$

The projective geometry on a line is the study of the properties which are preserved by the group of these transformations.

Note that if we used the field $\mathbb{C}$ instead of R, we would have got the Möbius transformation: the Riemann sphere is a projective complex line.

Instead of working with one number x to localize a point we can use so-called *homogeneous coordinates* (X, T) which are defined up to a non zero multiplicative constant. That means that if $T \neq 0$, then $x = \frac{X}{T}$ and if $T = 0$, then X has to be different from 0 and $\frac{X}{T} = \infty$. The formulae for expressing f become:

$$\begin{cases} X' &= \alpha X + \beta T \\ T' &= \gamma X + \delta T \end{cases}$$

Projective geometry in two dimensions

We add to the usual plane Π one projective line. A point can be an "old" point (x, y) and then we use homogeneous coordinates (X, Y, T) with $T \neq 0$ and $x = \frac{X}{T}$ and $y = \frac{Y}{T}$, or it can be a "new" point $(X, Y, 0)$ with $(X, Y) \neq (0, 0)$. The point $(X, Y, 0)$ can be viewed as a direction point common to all the lines parallel to the line

$$y = \frac{Y}{X}\, x,$$

see Figure 64.

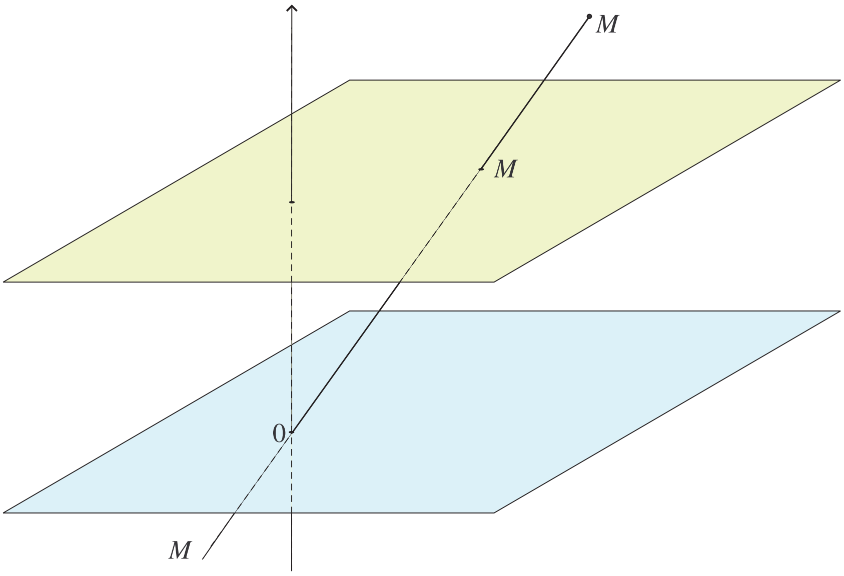

Figure 64: The projective plane

We can also define the projective plane as the set of one dimensional subspaces of a three dimensional linear space, for instance the universal covering space of Π or …a disc glued to the border of a Möbius strip, see Figure 65 !

With this concept it is easy to prove the Pappus's theorem or Desargues's theorem.

6.4.4 Geometry in 4 dimensions and more

Problem. Given a hypercube in a euclidean four dimensional space, find the intersection with a three dimensional space going through the center of the cube and orthogonal to a main diagonal.

We can define the cube by $-1 \leq x_i \leq 1$ for $i = 1, 2, 3, 4$. The three dimensional space Σ orthogonal to the diagonal through $(1, 1, 1, 1)$ has the equation:

$$x_1 + x_2 + x_3 + x_4 = 0$$

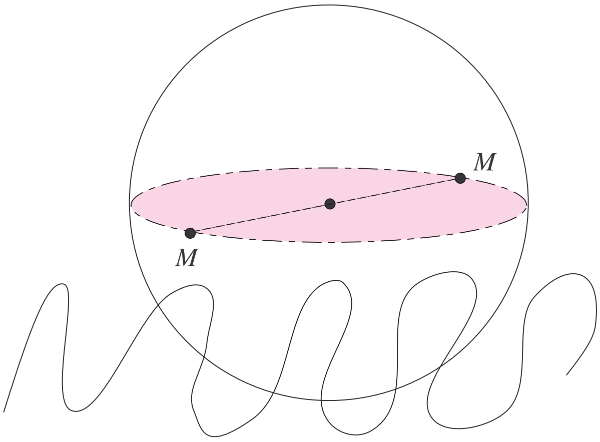

Figure 65: The Ball

Each face has the equation $x_i = \pm 1$. For instance the intersection of Σ with the face $x_1 = -1$ is:

$$\begin{cases} x_1 & & & & & = & -1 \\ x_2 & + & x_3 & + & x_4 & = & 1 \end{cases}$$

which is the triangle with summits $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. So we get a regular polyhedra with 8 triangular faces: it is a regular octahedra.

Penrose tilings and quasi-crystals

See pages from the net.

6.4.5 Non euclidean geometries

Carl Friedrich Gauss (1777-1855)

János Bolyai (1802-1860)

Nicolaï Ivanovitch Lobatchevsky (1752-1856)

Henri Poincaré (1854-1912)

Georg Bernhard Riemann (1821-1866)

6.4.6 The achievement of classical geometry

Felix Klein (1849-1925) Erlangen programm, 1872

David Hilbert (1862-1943) Grundlagen der Geometrie, 1899

6.4.7 More points on a segment than in a square !

Georg Cantor (1845-1918)

Theorem 6.4.3 Let E be a set and $P(E)$ the set of subsets of E. There is no surjective map from E on $P(E)$.

Proof. Let f be a surjective map from E on $P(E)$. Define

$$A := \{x \in E \mid x \notin f(x)\}.$$

Then $x \in A \Longleftrightarrow x \notin A$. ■

Remark 6.4.4 $\mathbb{R}$ and $P(\mathbb{N})$ have same cardinality, that is there is a bijection of one on the other.

Problem. Is there any set with strictly more elements than $\mathbb{N}$ and strictly less than $\mathbb{R}$?
Answer (Paul Cohen): It is as you want!

The Cantor set

Let t be a real number belonging to $[0,1]$. Write it in the base three:

$$t = 0{,}1102100212101212021021212010120120021\ldots$$

For some numbers (the rational numbers p/q where q is a power of three) there are two possibilities. Then we take the writing with the smallest amount of occurences of the numeral 1.

We choose $0{,}2120222222222\ldots$ rather than $0{,}2121000000000\ldots$

we choose $0{,}00200000\ldots$ rather than $0{,}0012222222\ldots$

Then we keep only the points having only the numerals 0 and 2 in their development. Thus we get the Cantor set C which has a length of

$$\lim \left(\frac{1}{3}\right)^n = 0.$$

The Peano curve

Giuseppe Peano (1858-1932)

Let t be a real number belonging to C:

$$t = 2\sum_{i=1}^{\infty} a_i 3^{-i},$$

we associate $(x(t), y(t))$ where:

$$\begin{cases} x(t) &= \displaystyle\sum_{i=1}^{\infty} a_{2i-1} 2^{-i} \\ x(t) &= \displaystyle\sum_{i=1}^{\infty} a_{2i} 2^{-i} \end{cases}$$

and for points in $[0,1] \setminus C$, we interpolate linearly. We get thus a continuous parametric curve which is surjective on the whole square $[0,1]^2$.